TI 2 Zusatzübung

18. AUG. 2011

5. a.)

$$I((Y,H),X) = H(X) - H(X|(Y,H))$$
$$\geq H(X) - H(X|Y)$$
$$= I(X,Y)$$

b.)

$$Y = HX + Z$$

$$C_{\text{no CSIT}} = \sup_{\{p(x) \mid E[X^2] \leq 1\}} I((Y,H),X)$$

channel state information transmitter

für jede Realisierung von H
wie AWGN Kanal $P = 1$
da $E[X^2] \leq 1$
$N = 1$
da $Z \sim N(0,1)$

$$= E_H\left[\frac{1}{2}\log\left(1 + \frac{P \cdot H^2}{N}\right)\right]$$
$$= E_H\left[\frac{1}{2}\log(1 + H^2)\right]$$
$$= P(H = \sqrt{0{,}5}) \cdot \frac{1}{2}\log(1 + 0{,}5) + P(H = \sqrt{1{,}5})\,\frac{1}{2}\log(1 + 1{,}5)$$
$$= \frac{1}{4}\left[\log\left(\frac{3}{2}\right) + \log\left(\frac{5}{2}\right)\right]$$
$$= \frac{1}{4}\left[\log\left(\frac{15}{4}\right)\right] \approx \underline{0{,}3304}$$

c.)

$$C_{CSIT} = \max_{\substack{0 \leq P_1 + P_2 \leq 2 \\ P_1, P_2 \geq 0}} \left\{\frac{1}{2}\frac{1}{2}\log\left(1 + \frac{P_1 H_1^2}{N}\right) + \frac{1}{2}\frac{1}{2}\log\left(1 + \frac{P_2 H_2^2}{N}\right)\right\}$$

$$H_1 = \sqrt{0{,}5} \qquad H_2 = \sqrt{1{,}5}$$

$$= \max_{\substack{c \le P_1 + P_2 \le 2 \\ P_1, P_2 \ge 0}} \frac{1}{4} \left\{ \log\left(1 + \frac{P_1}{2}\right) + \log\left(1 + P_2 \cdot \frac{3}{2}\right) \right\}$$

Es gilt Paralleler [redacted] Gaußkanal

$$C = \frac{1}{2} \sum_{i=1}^{2} \log\left(1 + \frac{(\nu - \lambda_i)^+}{\lambda_i}\right)$$

mit $\Sigma_Z = \Gamma \operatorname{diag}(\lambda_1, \lambda_2) \Gamma'$

$$\sum_{i=1}^{2} (\nu - \lambda_i)^+ = L$$

hier $\Sigma_Z = \begin{pmatrix} 2 & 0 \\ 0 & 2/3 \end{pmatrix}$

$$\underbrace{(\nu - 2)^+}_{P_1} + \underbrace{\left(\nu - \frac{2}{3}\right)^+}_{P_2} = 2$$

Test, ob $P_1, P_2 > 0$ als Lösung möglich

$$2\nu - 8/3 = 2 \Rightarrow \nu = \frac{1}{2} \cdot \left(2 + \frac{8}{3}\right) = \underline{\underline{7/3}}$$

dafür ist P_1, P_2 positiv [redacted]

$$P_1 = \frac{7}{3} - 2 = \underline{\frac{1}{3}} \qquad P_2 = \frac{7}{3} - \frac{2}{3} = \underline{\frac{5}{3}}$$

$$C_{S,T} = \frac{1}{4} \left\{ \log\left(1 + \frac{1}{2} \cdot \frac{1}{3}\right) + \log\left(1 + \frac{3}{2} \cdot \frac{5}{3}\right) \right\}$$

$$= \frac{1}{4} \left\{ \log\left(\frac{7}{6}\right) + \log\left(\frac{7}{2}\right) \right\}$$

$$= \frac{1}{4} \log\left(\frac{49}{12}\right) = 0{,}3517 \ \frac{\text{nat}}{\text{cu}}$$

Die Kanalrealisierung ...

T12 Zusatzübung

18. AUG. 2011

6. a.)

$\underline{z} = \mathcal{N}_2(\underline{0}, C)$

$$p_{\underline{z}}(\underline{z}) = \frac{1}{\sqrt{(2\pi)^2 \det(C)}} \exp\left(-\frac{1}{2}\underline{z}^T C^{-1} \underline{z}\right)$$

$$C = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

$$C^{-1} = \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} \begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{pmatrix}$$

$$-\frac{1}{2}\underline{z}^T C^{-1} \underline{z} = -\frac{1}{2} \frac{1(z_1\, z_2)}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} \begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$= -\frac{1}{2} \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} \left(\sigma_2^2 \cdot z_1^2 - 2\sigma_{12} z_1 z_2 + \sigma_1^2 z_2^2\right)$$

Vergleich mit $-\frac{2}{3}(z_1^2 - z_1 z_2 + z_2^2)$ Aufgabenstellung

$\Rightarrow \sigma_1^2 = \sigma_2^2 = 1$

$\sigma_{12} = \frac{1}{2}$

$C = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ z_1 und z_2 sind nicht s. u.

b.)

$$K = \frac{1}{2} \sum_{i=1}^{2} \log\left(1 + \frac{(\nu - \lambda_i)^+}{\lambda_i}\right)$$

mit $\sum_{i=1}^{2} (\nu - \lambda_i)^+ = L$

$\Sigma_2 = T \operatorname{diag}(\lambda_1, \lambda_2) T'$

Eigenwerte von C

~~...~~ $\det(C - \lambda I) = \det\begin{pmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & 1-\lambda \end{pmatrix} = 1 - 2\lambda + \lambda^2 - \frac{1}{4} \overset{!}{=} 0$

2

$$\Leftrightarrow \quad 0 = \lambda^2 - 2\lambda + \frac{3}{4}$$

$$\lambda_{1/2} = 1 \pm \sqrt{1 - \frac{3}{4}} = 1 \pm \frac{1}{2}$$

$$\lambda_1 = \frac{1}{2} \qquad \lambda_2 = \frac{3}{2}$$

water filling $\quad (\nu - \frac{3}{2})^+ + (\nu - \frac{1}{2})^+ = 5$

Test ob Leistungen beider Kanäle positiv sein können

$$2\nu - 2 = 5 \quad \Rightarrow \quad \underline{\underline{\nu = 3{,}5}}$$

$$K = \frac{1}{2}\log\left(1 + \frac{(3{,}5 - \frac{1}{2})^+}{1/2}\right) + \frac{1}{2}\log\left(1 + \frac{(3{,}5 - 3/2)^+}{3/2}\right)$$

$$= \frac{1}{2}\log(7) + \frac{1}{2}\log(7/3)$$

$$= \frac{1}{2}\log\left(\frac{49}{3}\right) \approx \underline{\underline{1{,}397}} \ \frac{\text{nat}}{\text{cu}}$$

c.)

$$\underline{x} \sim \mathcal{N}(\underline{0}, Q)$$

$$Q = T \cdot \operatorname{diag}\left((\nu - \lambda_i)^+\right) T^{-1}$$

Bestimmung von $T = (\underline{t}_1, \underline{t}_2)$

$$C \cdot \underline{t}_1 = \lambda_1 \underline{t}_1$$

$$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \cdot \begin{pmatrix} t_{11} \\ t_{12} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} t_{11} \\ t_{12} \end{pmatrix}$$

$$t_{11} + \frac{1}{2} t_{12} = \frac{1}{2} t_{11} \quad \Rightarrow \quad t_{12} = -t_{11}$$

$$\frac{1}{2} t_{11} + t_{12} = \frac{1}{2} t_{12} \qquad \underline{t}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

TI2 Zusatzübung

$$\begin{pmatrix}1 & 1/2\\ 1/2 & 1\end{pmatrix}\begin{pmatrix}t_{21}\\ t_{22}\end{pmatrix} = \frac{3}{2}\begin{pmatrix}t_{21}\\ t_{22}\end{pmatrix}$$

$$t_{21} + \frac{1}{2} t_{22} = \frac{3}{2} t_{21} \Rightarrow t_{21} = t_{22}$$

$$\Rightarrow \underline{t}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$$

$$Q = T \operatorname{diag}\left((\nu-\lambda_i)^+\right) T'$$

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}3{,}5-\frac{1}{2} & 0\\ 0 & 3{,}5-3/2\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}-3 & 3\\ 2 & 2\end{pmatrix}\frac{1}{\sqrt{2}} = \frac{1}{2}\begin{pmatrix}5 & -1\\ -1 & 5\end{pmatrix}$$

7.a)

3 Sende- und 3 Empfangsantennen

Kapazität des MIMO-Kanals

$$C = \sum_{i=1}^{t}\left[\log\left(\frac{\nu\gamma_i^2}{\sigma^2}\right)\right]^+$$

$$H^*H = U \operatorname{diag}(\gamma_1^2, \dots, \gamma_t^2)\, U^*$$

$$\sum_{i=1,\, \gamma_i>0}^{t}\left(\nu - \frac{\sigma^2}{\gamma_i^2}\right)^+ = L$$

$$H^*H = \begin{pmatrix}1 & 2 & 0\\ 0 & 0 & 2\\ 1 & -2 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\ 2 & 0 & -2\\ 0 & 2 & 0\end{pmatrix}$$

$$= \begin{pmatrix}5 & 0 & -3\\ 0 & 4 & 0\\ -3 & 0 & 5\end{pmatrix}$$

Eigenwerte

$$\det(H^*H - \lambda I) = \det\begin{pmatrix} 5-\lambda & 0 & -3 \\ 0 & 4-\lambda & 0 \\ -3 & 0 & 5-\lambda \end{pmatrix}$$

$$= (5-\lambda)^2(4-\lambda) - 9(4-\lambda)$$

$$= (4-\lambda)(25 - 10\lambda + \lambda^2 - 9)$$

$$= (4-\lambda)[\lambda^2 - 10\lambda + 16].$$

$$\lambda_{1/2} = 5 \pm \sqrt{25-16}$$
$$= 8 \vee 2$$

$$\lambda_1 = 2 \qquad \lambda_2 = 4 \qquad \lambda_3 = 8$$

Water <u>filling</u>

$$\left(\nu - \frac{16}{2}\right)^+ + \left(\nu - \frac{16}{4}\right)^+ + \left(\nu - \frac{16}{8}\right)^+ = 6$$

Teste, ob alle Leistungen f. alle Unterkanäle positiv sein ~~[illegible]~~ können.

$$3\nu - 8 - 4 - 2 = 6 \Rightarrow \nu = \frac{20}{3} = 6\tfrac{2}{3}$$

$\Rightarrow$ nicht möglich

Teste $\left(\nu - \frac{16}{4}\right)^+ + \left(\nu + \frac{16}{8}\right)^+ = 6$

$\Rightarrow \underline{\underline{\nu = 6}}$ möglich.

$$C = \sum_{i=1}^{3} \left[\log\left(\frac{\nu \gamma_i}{\sigma^2}\right)\right]^+$$
$$= \log\left(\frac{6 \cdot 4}{16}\right) + \log\left(\frac{6 \cdot 8}{16}\right)$$
$$= \log\left(\frac{3}{2}\right) + \log(3)$$
$$= \log\left(\frac{9}{2}\right) \approx 1{,}504 \ \frac{\text{nat}}{\text{cu}}$$

c) Kapazitäts erreichende Eingangsverteilung ist gegeben durch $X \sim SCN(\underline{0}, \underbrace{U \operatorname{diag}(\nu - \frac{\sigma^2}{\gamma_j}) \cdot U^*}_{Q})$

$$H^* H = U \operatorname{diag}(\gamma_1, \ldots, \gamma_3) U^* .$$

$$U = (\underline{u}_1 \ \underline{u}_2 \ \underline{u}_3)$$

$$H^* H \cdot \underline{u}_1 = \gamma_1 \underline{u}_1$$

$$\begin{pmatrix} 5 & 0 & -3 \\ 0 & 4 & 0 \\ -3 & 0 & 5 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \end{pmatrix} = 2 \begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \end{pmatrix} \qquad \underline{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}$$

$$\Rightarrow u_{12} = 0 \qquad \Rightarrow u_{13} = u_{11}$$

$$\begin{pmatrix} 5 & 0 & -3 \\ 0 & 4 & 0 \\ -3 & 0 & 5 \end{pmatrix} \begin{pmatrix} u_{21} \\ u_{22} \\ u_{23} \end{pmatrix} = 4 \begin{pmatrix} u_{21} \\ u_{22} \\ u_{23} \end{pmatrix} \Rightarrow \underline{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 0 & -3 \\ 0 & 4 & 0 \\ -3 & 0 & 5 \end{pmatrix} \begin{pmatrix} u_{31} \\ u_{32} \\ u_{33} \end{pmatrix} = 8 \begin{pmatrix} u_{31} \\ u_{32} \\ u_{33} \end{pmatrix} \Rightarrow \underline{u}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\underline{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} (6-16/2)^+ & 0 & 0 \\ 0 & (6-\frac{16}{4})^+ & 0 \\ 0 & 0 & (6-\frac{16}{8})^+ \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 4 & 0 & -4 \\ 0 & 1 & 4 \\ -4 & 0 & 4 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2 & 0 & -2 \\ 0 & 1/2 & 2 \\ -2 & 0 & 2 \end{pmatrix} = Q}}$$

<u>8.)</u> <u>a)</u> $N_{A/B}$ Anzahl Server A/B

min ① $N_A \cdot 10\,000 + N_B \cdot 5000$ (Kosten)

s.d. ② $N_A \cdot 1000 + N_B \cdot 600 \geq 5000$

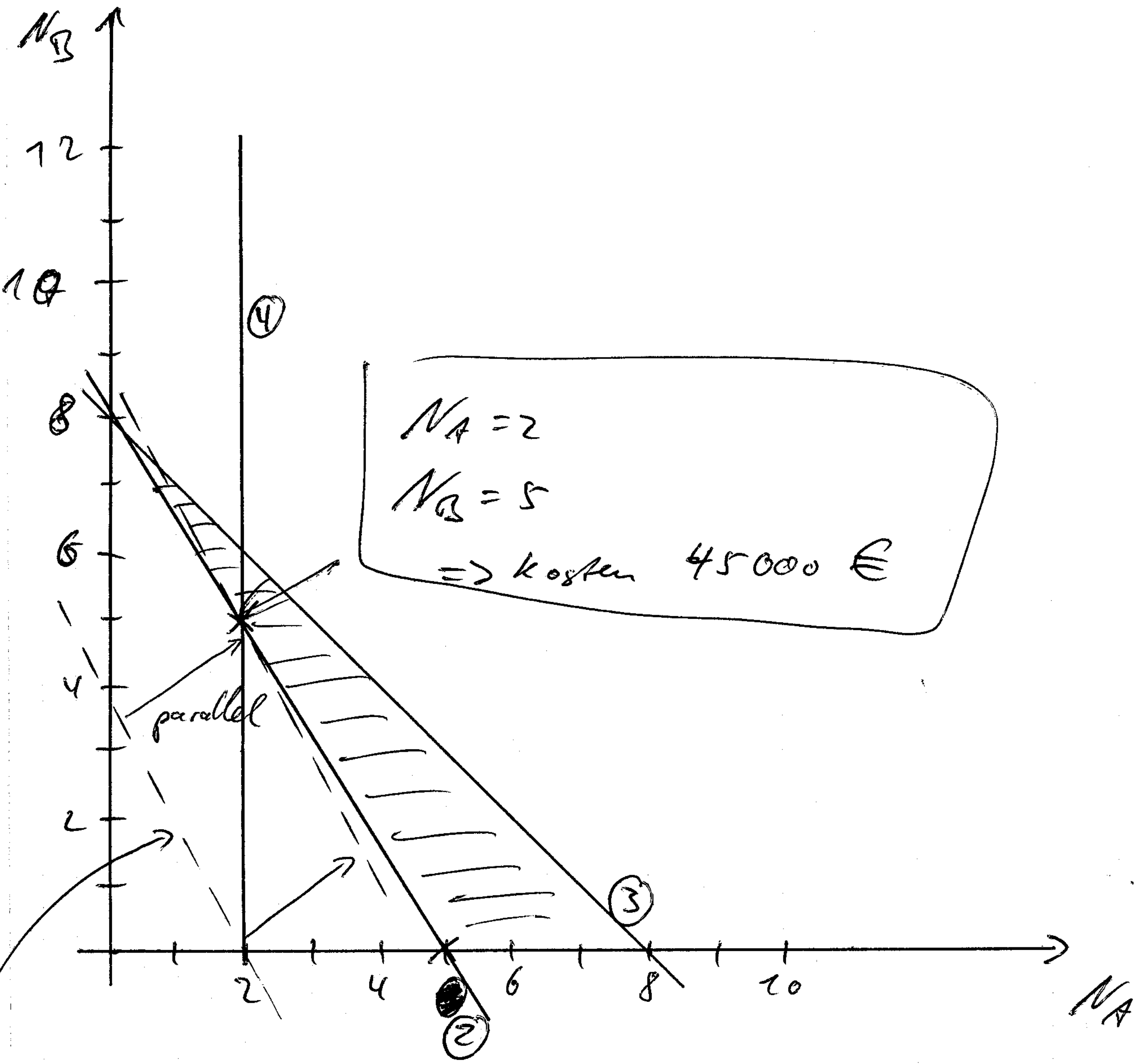
③ $N_A + N_B \leq 8$

④ $N_A \geq 2$

⑤ $N_A, N_B \in \mathbb{N}_0$

V 1 2 Zusatzübung

8. b.)



② <=> $N_A \geq 5 - 0{,}6 N_B$

① $N_A \cdot 10\,000 + N_B \cdot 5000 = K$

$$2 N_A + N_B = \frac{K}{5000}$$

K=20.000