B45

①

HöMa ÜG ⑨ 22.06.09
B-Teil

d) Löse: $y' \sin y = y \ln y$ mit $y\left(\frac{\pi}{2}\right) = 2$

$$\int \frac{y'\,dy}{y \ln y} = \int \frac{1\,dx}{\sin x}$$

$$\Leftrightarrow \ln(\ln(y)) = \ln\sqrt{\tan\left(\tfrac{x}{2}\right)} + C$$

$$\Leftrightarrow \ln(y) = \tilde{C}\sqrt{\tan\left(\tfrac{x}{2}\right)}$$

$$\Leftrightarrow y = e^{\tilde{C}\sqrt{\tan\left(\frac{x}{2}\right)}}$$

$$y\left(\tfrac{\pi}{2}\right) = e^{\tilde{C}\sqrt{\tan\left(\frac{\pi}{4}\right)}} = e^{\tilde{C}} \overset{!}{=} 2$$

$$\Rightarrow \tilde{C} = \ln 2$$

$$\Rightarrow y(x) = e^{\ln(2)\sqrt{\tan\left(\frac{x}{2}\right)}} = 2^{\sqrt{\tan\left(\frac{x}{2}\right)}}$$

Nebenrechnung:

$$\int \frac{y'\,dy}{y \ln y} = \int \frac{du}{u} = \ln u = \ln(\ln y)$$

$$\ln y = u$$

$$\frac{1}{y} y'\,dy = du$$

$$\int \frac{1\,dx}{\sin(x)} = \int \frac{dt}{2t} = \frac{1}{2}\ln t = \ln\sqrt{t} = \ln\sqrt{\tan\left(\tfrac{x}{2}\right)}$$

HWM: $\sin(x) = \frac{2t}{1+t^2}$, $dx = \frac{1}{1+t^2}$

$$\tan\left(\tfrac{x}{2}\right) = t$$

B45 b) $v'(t) = v(t)\sin(t) + t^2 e^{-\cos(t)}$ (1) mit $v(0) = 1$

Homogen: $v_h'(t) = v_h(t)\sin(t) \Leftrightarrow \frac{v_h'}{v_h} = \sin(t)$

$$\Leftrightarrow \ln|v_h| = -\cos(t) + C \Leftrightarrow v_h = \tilde{C} e^{-\cos(t)}$$

spezielle Lsg.: $v(t) = \tilde{C} + e^{-\cos(t)}$, $v' = \tilde{C}' e^{-\cos(t)} \cdot \sin(t)$ in (1) einsetzen:

$$\tilde{C}' \cancel{e^{-\cos(t)}} + \cancel{\tilde{C}} \cancel{e^{-\cos(t)}} \cdot \cancel{\sin(t)} = \cancel{\tilde{C}} e^{-\cos(t)} \cdot \cancel{\sin(t)} + t^2 \cancel{e^{-\cos(t)}}$$

$$\Leftrightarrow \tilde{C}' = t^2 \Rightarrow \tilde{C} = \frac{1}{3}t^3 + C_1$$

$$\Rightarrow v(t) = \frac{1}{3}t^3 e^{-\cos(t)} + C_1 e^{-\cos(t)}$$

$$v(0) = C_1 \frac{1}{e} \overset{!}{=} 1 \Rightarrow C_1 = e \Rightarrow v(t) = \frac{1}{3}t^3 e^{-\cos(t)} + e^{1-\cos(t)}$$

B45 a)

$$x\, u'(x) \ln(x) + u^2(x)\, x - u(x) = 0 \quad \Big| \cdot \frac{1}{x \ln(x)}$$

$$\Leftrightarrow \quad u' - \frac{u}{x \ln(x)} = \frac{-u^2}{\ln(x)} \quad (1)$$

Typ: $u'(x) + a(x)\, u(x) = b(x)\, u^{\alpha}(x)$ (Bernoulli-DGL)

Ansatz: $u(x) = v^{\beta}(x)$, $\beta = \dfrac{1}{1-\cancel{\alpha}}$

$$\beta = -1 \Rightarrow u(x) = v^{-1}(x)$$

$u'(x) = -v^{-2}(x) \cdot v'(x)$ in (1) einsetzen:

$$-v^{-2}(x)\, v'(x) - \frac{v^{-1}(x)}{x \ln(x)} = \frac{-v^{-2}(x)}{\ln x}$$

$$\Leftrightarrow v' + \frac{v}{x \ln(x)} = \frac{1}{\cancel{x}\ln(x)} \quad \ldots\ldots \quad \Leftrightarrow \quad v(x) = \frac{x+c}{\ln(x)}$$

$$\Rightarrow u(x) = \frac{\ln(x)}{x+c}$$

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c) $y(x) = -\sqrt{2 \ln\left(\dfrac{1+e^x}{1+e}\right) + 1}$

e) $y(x) = \dfrac{1}{\ln(x) + 1}$

B 44

③

a) $u''(x) + 6u'(x) + 9u(x) = 0$

Ansatz: $u(x) \sim e^{\lambda x}$ $\quad u'(x) \sim \lambda e^{\lambda x}$ $\quad u''(x) \sim \lambda^2 e^{\lambda x}$

$$\lambda^2 \cancel{e^{\lambda x}} + 6\lambda \cancel{e^{\lambda x}} + 9 \cancel{e^{\lambda x}} = 0$$

$\Leftrightarrow \lambda^2 + 6\lambda + 9 = 0$

$\Leftrightarrow (\lambda+3)^2 = 0 \qquad \lambda_{1,2} = -3$

$$\boxed{u(x) = A e^{\lambda_1 x} + B e^{\lambda_2 x} \quad \text{nur: für } \lambda_1 \neq \lambda_2}$$

$$u(x) = A e^{\lambda x} + x B e^{\lambda x}$$

$$= A e^{-3x} + x B e^{-3x}$$

$u'' + 6u' + 9u = 0$

$u = v$

$u' = w$

$v' = w$

$w' = -6w - 9v$

$$\begin{pmatrix} v' \\ w' \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 \\ -9 & -6 \end{pmatrix}}^{A} \begin{pmatrix} v \\ w \end{pmatrix}$$

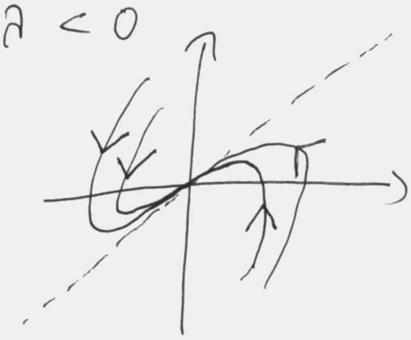
EW von A:

$|A - \lambda E| \overset{!}{=} 0 \Rightarrow \lambda_{1,2} = -3 \quad \boxed{\{ \alpha v_1 e^{\lambda_1 x} \ldots}$

Skr. S. 78 Phasendiagramm

λ doppelter EW

$\lambda < 0$



B44

④

b) $u'' - 4u' + 13u = 0$ Ansatz: $u \sim e^{\lambda x}$

$\lambda^2 - 4\lambda + 13 = 0 \qquad \lambda_1 = 2 + 3i \qquad \lambda_2 = 2 - 3i$

$u(x) = A e^{2x} e^{3ix} + B e^{2x} e^{-3ix}$

$= e^{2x}\left[A\cos(3x) + Ai\sin(3x) + B\cos(3x) - Bi\sin(3x)\right]$

$= e^{2x}\left[(A+B)\cos(3x) + i(A-B)\sin(3x)\right]$

$\boxed{B = A^*}$ kompl. konj.

$A + B = A + A^* = 2\,\mathrm{Re}(A)$

$A - B = A - A^* = 2i\,\mathrm{Im}(A)$

$\left.\begin{array}{l} u = v \\ u' = w \end{array}\right\}$ $EW(A) = 2 \pm 3i$

B43 a) $u' = Au \qquad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} \qquad u(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

EW von A:

$\begin{vmatrix} 1-\lambda & 2 \\ -3 & -4-\lambda \end{vmatrix} = (\lambda+1)(\lambda+2) \Rightarrow \lambda_1 = -1, \lambda_2 = -2$

EV zu $\lambda_1 = -1$:

$(A - \lambda_1 E)\underline{v} = 0 \quad \begin{pmatrix} 2 & 2 \\ -3 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \rightarrow 2v_1 = -2v_2$

$\Leftrightarrow v_1 = -v_2 \Rightarrow v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

EV zu $\lambda_2 = -2$:

$\begin{pmatrix} 3 & 2 \\ -3 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 2 \\ 0 & 0 \end{pmatrix} \rightarrow 3w_1 = -2w_2 \Rightarrow w = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

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$$\Rightarrow u \in \left\{ \alpha v e^{\lambda_1 x} + \beta w e^{\lambda_2 x}, \alpha, \beta \in \mathbb{R} \right\}$$

$$u(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-x} + \beta \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{-2x}$$

$$\Rightarrow \alpha = 1, \ \beta = -1$$

$$\Rightarrow u(x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-x} - \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{-2x}$$

B43 c) $u' = A \cdot u \quad A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ u(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

EW

$$|A - \lambda E| = (1-\lambda)(\lambda - 1)(\lambda + 1) \Rightarrow \lambda = -1$$

$\mu = 1$ alg. Vielfachheit 2

EV zu $\lambda = -1$: $(A - \lambda E)\underline{v} = 0 \Rightarrow \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

$$v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

EV zu $\mu = 1$: $(A - \mu E)\underline{w} = 0 \Rightarrow \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$ $\begin{array}{l} \to w_2 = w \\ \to w_1 = w \end{array}$

$$\underline{w}^1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

geometrische Vielfachheit von μ ist 1

$\Rightarrow$ HV 2. Stufe bestimmen

$$(A-\mu B)\,\underline{w}^2 = \underline{w}^1$$

$$\left(\begin{array}{ccc|c} 0 & 1 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 0 & 1 & -1 & 1 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left.\begin{array}{l} w_2^2 = 1 + w_3^2 \\ w_1^2 = 2 + w_3^2 \end{array}\right\} \quad w^2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\lambda x = -x$ $\qquad$ $\mu x = x$

$$u(x) = \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-x} + \beta \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{x} + \gamma \left[\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right] e^{x}$$

$$u(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-x} + \beta \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{x} + \gamma \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{x} + \gamma x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{x}$$

$$\Rightarrow \alpha = \frac{1}{4}, \quad \beta = \frac{1}{4}, \quad \gamma = \frac{1}{2}$$

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b) $$u(x) = \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} - x \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right] e^{x}$$