P21

Bestimmung von f_X, f_Y: (Randdichten)

a) 1) $x < 0$:

$$f_X(x) \underset{\text{Def.}}{=} \int_{-\infty}^{\infty} \underbrace{f_{X,Y}(x,y)}_{=0 \text{ für } x<0} \, dy = 0$$

$x \geq 0$:

$$f_X(x) \overset{\text{Def.}}{=} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy$$

$$= \int_2^4 \tfrac{1}{2} e^{-x} \, dy = e^{-x}$$

$$\Rightarrow f_X(x) = \begin{cases} 0 & , x < 0 \\ e^{-x} & , x \geq 0 \end{cases}$$

2) $y \in \mathbb{R} \setminus [2,4]$

$$f_Y(y) \overset{\text{Def.}}{=} \int_{-\infty}^{\infty} \underbrace{f_{X,Y}(x,y)}_{=0 \text{ für } y \in \mathbb{R} \setminus [2,4]} \, dx = 0$$

$y \in [2,4]$

$$f_Y(y) \overset{\text{Def.}}{=} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx$$

$$= \int_0^{\infty} \tfrac{1}{2} e^{-x} \, dx = \frac{1}{2}$$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{2} & , y \in [2,4] \\ 0 & , y \notin [2,4] \end{cases}$$

b) $$E(X) \overset{\text{Def.}}{=} \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

$$= \int_{0}^{\infty} x \cdot e^{-x}\,dx \overset{\text{Part. Int.}}{=} \dots = 1$$

$$E(Y) \overset{\text{Def.}}{=} \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy = \int_{2}^{4} y \cdot \frac{1}{2}\,dy = \dots = 3$$

$$E(X^2) \overset{\substack{\text{Trafosatz}\\ \text{für EWerte}}}{=} \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_{0}^{-\infty} x^2 e^{-x}\,dx$$

$$\overset{\text{part. Int.}}{=} \dots = 2$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy = \int_{2}^{4} \frac{1}{2} y^2\,dy = \dots = \frac{28}{3}$$

Verschiebungs-Satz $\Rightarrow$

$$Var(X) = E(X^2) - (E(X))^2 = 2 - 1^2 = 1$$

$$Var(Y) = E(Y^2) - (E(Y))^2 = \frac{28}{3} - 3^2 = \frac{1}{3}$$

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c) $$E(X,Y) \overset{\text{Def.}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{(X,Y)}(x,y)\,dx\,dy$$

$$= \int_{2}^{4}\int_{0}^{\infty} x\,y\,\frac{1}{2}\,e^{-x}\,dx\,dy = \int_{2}^{4} \frac{1}{2} y \underbrace{\int_{0}^{\infty} x e^{-x}\,dx}_{=1}\,dy$$

$$= \dots = 3$$

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d) $\operatorname{cov}(X,Y) = E(X,Y) - E(X)\,E(Y) = 3 - 1 \cdot 3 = 0$

$$COR(X,Y) = \frac{cov(X,Y)}{\sqrt{VAR(X) \cdot Var(Y)}} = \frac{0}{\sqrt{1 \cdot \frac{1}{3}}} = 0$$

$\Rightarrow$ X, Y unkorreliert

Allgemein: unkorreliert $\not\Rightarrow$ stoch. unabh.

$\Leftarrow$

Frage: X, Y st. unabh.?

$\rightarrow$ gilt $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$?

Fall 1: $x \in [0, \infty)$, $y \in [2,4]$

$$f_{X,Y}(x,y) = \frac{1}{2} e^{-x} = f_X(x) \cdot f_Y(y)$$

Fall 2: $x \in (-\infty, 0)$ oder $y \in \mathbb{R} \setminus [2,4]$

$$f_{X,Y}(x,y) = 0 = f_X(x) \cdot f_Y(y)$$

$\Rightarrow$ X, Y stoch. unabh.

P22

x_i: Anzahl Münzen in i-ter Rolle

für $i \in \{1, \dots, 1000\}$

↳ Annahme: stoch. unabh., identisch verteilte

(kurz iid) ZV $x_1, \dots, x_{1000}$

k	48	49	50	51
$P(x_i = k)$	0,1	0,2	0,6	0,1

$$\Rightarrow \quad \mu = E(x_i) = \sum_{k=48}^{51} k \cdot P(x_i = k)$$

$$= 48 \cdot 0{,}1 + 49 \cdot 0{,}2 + 50 \cdot 0{,}6 + 51 \cdot 0{,}1 = \underline{\underline{49{,}7}}$$

$$\sigma^2 = Var(x_i) = E(x_i^2) - \left(E(x_i)\right)^2 = \sum_{k=48}^{51} k^2 P(x_i = k) - 49{,}7$$

$$= \dots = 0{,}61$$

für $S_{1000} = \sum_{i=1}^{1000} x_i$:

$$E(S_{1000}) = 1000 \cdot \mu = 49.700$$

$$Var(S_{1000}) \underset{\text{st. unabh.}}{=} 1000 \cdot \sigma^2 = 610$$

Allgemein:

$$E(S_n) = n \cdot \mu$$

$$Var(S_n) = n \cdot \sigma^2$$

P22 (5)

a) $P(S_{1000} \geq 49.650) = 1 - \underbrace{P(S_{1000} < 49650)}_{\underset{S_n \in \mathbb{N}_0}{=} P(S_{1000} \leq 49649)}$

$= 1 - P\Big(\underbrace{\frac{S_{1000} - 49700}{\sqrt{610}}}_{\sim \omega(0,1)} \leq \frac{49649 - 49700}{\sqrt{610}}\Big)$

$= 1 - \phi\left(\frac{49649 - 49700}{\sqrt{610}}\right) = -2{,}06$

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b) $P(49670 \leq S_{1000} \leq 49736)$

$= P(S_{1000} \leq 49730) - \underbrace{P(S_{1000} < 49670)}_{\underset{S_{1000} \in \mathbb{N}_0}{=} P(S_{1000} \leq 49669)}$

$= \ldots = \phi(1{,}21) - \underbrace{\phi(-1{,}26)}_{1 - \phi(1{,}26)}$

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c) ges. $P(S_n > 50n)$, $n \in \{10, 100, 1000\}$

$P(S_n > 50n) = 1 - P(S_n \leq 50n)$

$= \ldots = 1 - \phi\left(\frac{0{,}3n}{\sqrt{0{,}61n}}\right)$

P 23

$X \sim \mathrm{Poi}(\lambda)$, $\lambda > 0$, d.h.

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k \in \mathbb{N}_0$$

a) $$g_X(t) \overset{\text{Def.}}{=} E(t^X) \underset{\substack{\text{Trafoformel} \\ \text{E Werte}}}{=} \sum_{k=0}^{\infty} t^k \cdot P(X = k)$$

$$= \sum_{k=0}^{\infty} t^k \cdot \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \underbrace{\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}}_{= e^{\lambda t}}$$

Reihendarstellung e - Fkt.

$$= e^{\lambda(t-1)}, \quad t \in [-1, 1]$$

b) $$E(X) = g_X^{(1)}(1) = \lambda e^{\lambda(1-1)}$$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = g_X^{(2)}(1) + g_X^{(1)}(1) - \left(g_X^{(1)}(1)\right)^2$$

$$g_X^{(1)}(1) = E(X)$$

$$g_X^{(2)}(1) = E\left(X(X-1)\right) = E(X^2) - E(X)$$