21.04.11

TI: Übung 2

Aufg. 1

$$I(X;Y,Z) = H(Y,Z) - H(Y,Z|X)$$

$$= H(Y|Z) + H(Z) - H(Y|Z,X) - H(Z|X)$$

$$= H(Z) - H(Z|X) + H(Y|Z) - H(Y|Z,X)$$

$$= I(Z;X) + I(Y;X|Z)$$

$$= I(Z;X) + I(X;Y|Z)$$

Aufg. 2

$$\underline{Y} = A\underline{X} + \underline{b}$$

$$H(A\underline{X} + \underline{b}) = H(\underline{Y})$$

Transformationssatz

$$f_Y(y) = \frac{1}{|det(J)|} \cdot f_X(x = A^{-1}(\underline{y} - \underline{b}))$$

$$J = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{pmatrix}$$

$$\underline{Y} = T(\underline{X})$$

hier: $J = A \Rightarrow det(J) = det(A)$

$$\Rightarrow f_Y(\underline{y}) = \frac{1}{|det(A)|} f_X(\underline{x} = A^{-1}(\underline{y} - \underline{b}))$$

21.04.11

$$H(\underline{Y}) = -\int f_Y(\underline{y}) \cdot \log(f_Y(\underline{y}))\, d\underline{y}$$

$$\Gamma$$

$$\int_G f(y_1,y_2)\, dy_1\, dy_2 = \int_{G^*} f(T(x_1,x_2)) \cdot \left|\det\begin{pmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_2}{\partial x_2} \\ \frac{\partial T_2}{\partial x_1} & \frac{\partial T_2}{\partial x_2} \end{pmatrix}\right| dx_1\, dx_2$$

$$d\underline{y} = |\det(A)| \cdot d\underline{x}$$

$$\Rightarrow H(\underline{Y}) = -\int \frac{1}{|\det(A)|} f_X(\underline{x} = A^{-1}(\underline{y}-\underline{b})) \cdot \log\left(\frac{1}{|\det(A)|} f_X(\underline{x} = A^{-1}(\underline{y}-\underline{b}))\right)$$

$$\cdot |\det(A)|\, d\underline{x}$$

$$= \int \frac{1}{|\det(A)|} f_X(\underline{x} = A^{-1}(\underline{y}-\underline{b})) \cdot \log(|\det(A)|)\, |\det(A)|\, d\underline{x}$$

$$- \int \frac{1}{|\det(A)|} f_X(\underline{x} = A^{-1}(\underline{y}-\underline{b})) \cdot \log(f_X(\underline{x} = A^{-1}(\underline{y}-\underline{b}))\, |\det(A)|$$

$$= \log(|\det(A)|) - \int f_X(\underline{x}) \cdot \log(f_X(\underline{x}))\, d\underline{x}$$

$$= \log(|\det(A)|) + H(\underline{X})$$

<u>Aufg. 3</u>

a) $$tr(A \cdot B) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} \cdot b_{ki}$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ki}\, a_{ik} = tr(B \cdot A)$$

b) $$E(tr(X)) = E\left(\sum_{i=1}^{n} x_{ii}\right) = \sum_{i=1}^{n} E(x_{ii}) = tr(E(X))$$

21.04.11

Aufg. 4

a) $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{(x-\mu_1)^2}{2\sigma^2}\right)$

$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{(x-\mu_2)^2}{2\sigma^2}\right)$

$$D(f_X \| f_Y(x)) = \int_{-\infty}^{\infty} f_X(x) \cdot \log\left(\frac{f_X(x)}{f_Y(x)}\right) dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot \log\left(\frac{\cancel{\frac{1}{\sqrt{2\pi\sigma^2}}} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma^2}\right)}{\cancel{\frac{1}{\sqrt{2\pi\sigma^2}}} \exp\left(-\frac{(x-\mu_2)^2}{2\sigma^2}\right)}\right) dx$$

$$= \log(e) \int_{-\infty}^{\infty} f_X(x) \left[\frac{(x-\mu_2)^2}{2\sigma^2} - \frac{(x-\mu_1)^2}{2\sigma^2}\right] dx$$

$$= \log(e) \int_{-\infty}^{\infty} f_X(x) \frac{(2\mu_1 - 2\mu_2)x + \mu_2^2 - \mu_1^2}{2\sigma^2} dx$$

$$= \frac{\log(e)}{2\sigma^2} \left[(\mu_2^2 - \mu_1^2) \int_{-\infty}^{\infty} f_X(x) dx + 2(\mu_1 - \mu_2) \int_{-\infty}^{\infty} x \cdot f_X(x) dx\right]$$

$$= \frac{\log(e)}{2\sigma^2} \left[(\mu_2^2 - \mu_1^2) + 2(\mu_1 - \mu_2) \cdot \mu_1\right]$$

$$= \frac{\log(e)}{2\sigma^2} (\mu_1 - \mu_2)^2$$

21.04.11

5) $f_x(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{x^2}{2\sigma_1^2}\right)$

$f_y(y) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{y^2}{2\sigma_2^2}\right)$

$$D(f_x(x) \| f_y(x)) = \int_{-\infty}^{\infty} f_x(x) \cdot \log\left(\frac{f_x(x)}{f_y(x)}\right) dx$$

$$= \int_{-\infty}^{\infty} f_x(x) \log \frac{\frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{x^2}{2\sigma_1^2}\right)}{\frac{1}{\sqrt{2\pi}\sigma_2} \exp\left(-\frac{x^2}{2\sigma_2^2}\right)} dx$$

$$= \int_{-\infty}^{\infty} f_x(x) \cdot \log\left[\frac{\sigma_2}{\sigma_1} \cdot \exp\left(\frac{x^2}{2\sigma_2^2} - \frac{x^2}{2\sigma_1^2}\right)\right] dx$$

$\log\left(\frac{\sigma_2}{\sigma_1}\right)$

$$= \int_{-\infty}^{\infty} f_x(x) dx + \int_{-\infty}^{\infty} f_x(x) \cdot \log\left[\exp\left(\frac{x^2}{2\sigma_2} - \frac{x^2}{2\sigma_1^2}\right)\right] dx$$

$$= \log\left(\frac{\sigma_2}{\sigma_1}\right) + \log(e) \int_{-\infty}^{\infty} f_x(x) \left(\frac{x^2}{2\sigma_2^2} - \frac{x^2}{2\sigma_1^2}\right) dx$$

$$= \log\left(\frac{\sigma_2}{\sigma_1}\right) + \log(e) \left(\frac{1}{2\sigma_2^2} - \frac{1}{2\sigma_1^2}\right) \underbrace{\int_{-\infty}^{\infty} x^2 \cdot f_x(x) dx}_{= \sigma_1^2}$$

$$= \log\left(\frac{\sigma_2}{\sigma_1}\right) + \log(e)\, \sigma_1^2 \left(\frac{1}{2\sigma_2^2} - \frac{1}{2\sigma_1^2}\right)$$

Wegen 5) => KL-Distanz <u>nicht</u> symmetrisch!