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II: Übung 3

Aufg. 1

$y = x + z \qquad x \in \{ A, -A \} \qquad$ BPSK

$z \sim N(0,1)$

ges. $I(x,y)$

a) z.z. $f_{Y|X}(y \mid X=x) = f_Z(y-x)$

Transformationssatz

$T(z) = z + x \quad \to \quad T^{-1}(y) = y - x$

$$f_{Y|X}(y \mid X=x) = \left| \frac{\partial T^{-1}}{\partial y} \right| \cdot f_Z(T^{-1}(y))$$

$$= 1 \cdot f_Z(y-x) = f_Z(y-x)$$

b) $$H(Y \mid X) = \sum_{i=1}^{2} \underbrace{P(X = x_i)}_{=1/2} \cdot H(Y \mid X = x_i)$$

$$= + \sum_{i=1}^{2} \frac{1}{2} \left(- \int_{-\infty}^{\infty} f_{Y|X}(y \mid X=x_i) \cdot \log f_{Y|X}(y \mid X=x_i) \, dy \right)$$

$$= - \sum_{i=1}^{2} \frac{1}{2} \int_{-\infty}^{\infty} f_Z(y - x_i) \cdot \log f_Z(y - x_i) \, dy$$

$$= - \sum_{i=1}^{2} \frac{1}{2} \int_{-\infty}^{\infty} f_Z(z) \log f_Z(z) \, dz = H(Z)$$

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c) $f_Y(y) = \sum_{i=1}^{2} f_{Y|X}(y \mid X = x_i) \cdot P(X = x_i)$

$= \sum_{i=1}^{2} \frac{1}{2} f_{Y|X}(y \mid X = x_i)$

$= \frac{1}{2} \sum_{i=1}^{2} \underbrace{f_Z(y - x_i)}_{\varphi_i}$

d) z.z. $I(X;Y) = D\left(\varphi_1 \,\middle\|\, \frac{1}{2} \sum_{i=1}^{2} \varphi_i \right)$

$I(X;Y) = H(Y) - H(Y|X)$

$= H(Y) - H(Z)$

$\overset{c)}{=} - \int_{-\infty}^{\infty} \frac{1}{2} \sum_{i=1}^{2} \underbrace{f_Z(y - x_i)}_{\varphi_i} \cdot \log\left(\frac{1}{2} \sum_{i=1}^{2} \underbrace{f_Z(y - x_i)}_{\varphi_i} \right) dy$

$+ \sum_{i=1}^{2} \frac{1}{2} \int_{-\infty}^{\infty} \underbrace{f_Z(y - x_i)}_{\varphi_i} \cdot \log \underbrace{f_Z(y - x_i)}_{\varphi_i} \, dy$

$= \frac{1}{2} \sum_{j=1}^{2} \int_{-\infty}^{\infty} \varphi_j \cdot \log\left(\frac{\varphi_j}{1/2 \sum_{i=1}^{2} \varphi_i} \right) dy$

$\overset{\text{Symmetrie } (*)}{=} \int_{-\infty}^{\infty} \varphi_1 \cdot \log\left(\frac{\varphi_1}{\frac{1}{2} \sum_{i=1}^{2} \varphi_i} \right) dy$

$= D\left(\varphi_1 \,\middle\|\, \frac{1}{2} \sum_{i=1}^{2} \varphi_i \right) = D\left(\varphi_2 \,\middle\|\, \frac{1}{2} \sum_{i=1}^{2} \varphi_i \right)$

$$(*)\quad \int_{-\infty}^{\infty} \varphi_1 \log\left(\frac{\varphi_1}{\frac{1}{2}\sum_{i=1}^{2}\varphi_i}\right) dy$$

$$= \int_{-\infty}^{\infty} f_Z(y-\sqrt{P}) \cdot \log\left(\frac{f_Z(y-\sqrt{P})}{\frac{1}{2}\left(f_Z(y-\sqrt{P}) + f_Z(y+\sqrt{P})\right)}\right) dy$$

$$\overset{x=-y}{=} -\int_{+\infty}^{-\infty} f_Z(-x-\sqrt{P}) \log\left(\frac{f_Z(-x-\sqrt{P})}{\frac{1}{2}\left(f_Z(-x-\sqrt{P}) + f_Z(-x+\sqrt{P})\right)}\right) dx$$

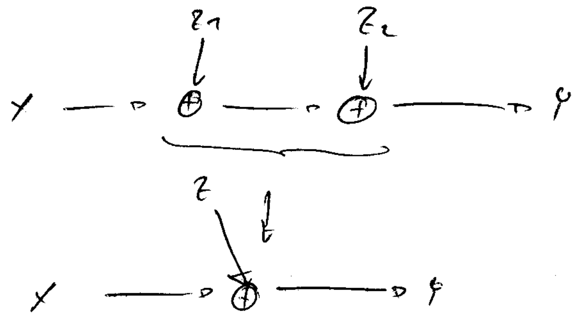
$Z \sim \mathcal{N}(0,1) \Rightarrow f_Z(z) = f_Z(-z)$

$$\overset{*}{=} \int_{-\infty}^{\infty} f_Z(x+\sqrt{P}) \cdot \log\left(\frac{f_Z(x+\sqrt{P})}{\frac{1}{2}\left(f_Z(x+\sqrt{P}) + f_Z(x-\sqrt{P})\right)}\right) dx$$

$$= \int_{-\infty}^{\infty} \varphi_2 \log\left(\frac{\varphi_2}{\frac{1}{2}\sum_{i=1}^{2}\varphi_i}\right) dy$$

Aufg. 2

a) $Y = X + Z_1 + Z_2$ $\qquad Z_1 \sim \mathcal{N}(0,\sigma_1^2)$, $Z_2 \sim \mathcal{N}(0,\sigma_2^2)$ s.u.



$Z = Z_1 + Z_2 \Rightarrow Z \sim \mathcal{N}(0, \sigma_1^2 + \sigma_2^2)$

$\hookrightarrow Y = X + Z_1 + Z_2 = X + Z$ mit X, Z s.u.

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$\rightarrow$ reelle Gaußkanal (Prob. 4.2.1)

$$C = \frac{1}{2} \ln\left(1 + \frac{L}{\sigma_1^2 + \sigma_2^2}\right)$$

Kapazität wird angenommen für $X \sim N(0, L)$

b) $\sigma_1^2 = 2 \quad \sigma_2^2 = 3$

$$C \overset{!}{=} 1 = \frac{1}{2} \ln\left(1 + \frac{L}{5}\right)$$

$$\Leftrightarrow L = (e^2 - 1) \cdot 5 \approx 31{,}95$$