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Aufgabe 1)

Relevante Ereignisse:

N = "Vorliegen Nutzsignal"

R = "Vorliegen Rauschen"

A = "Anzeige Nutzsignal"

$P(N) = 0.3, \quad P(R) = 0.7$

$P(A|N) = p_d = 0.9$

$P(A|R) = p_f = 0.2$

Gesucht:

$$P(N|A) = \frac{P(N \cap A)}{P(A)}$$

- $P(A) = P(A|N) \cdot P(N) + P(A|R) \cdot P(R)$

  $\Rightarrow P(A) = 0.9 \cdot 0.3 + 0.2 \cdot 0.7 = 0.41$

- $P(N|A) = \dfrac{P(N \cap A)}{P(A)} = \dfrac{P(A|N) \cdot P(N)}{P(A)}$

  $\Rightarrow P(N|A) = \dfrac{0.9 \cdot 0.3}{0.41} = 0.6585$

Aufgabe 2)

$$f_X(x_1, x_2) = \frac{1}{\pi\sqrt{3}} \cdot e^{-\frac{2}{3}(x_1^2 - x_1 x_2 + x_2^2)}$$

$$f_X(x_1, x_2) = \underbrace{\frac{1}{(2\pi)^{n/2} |C|^{\frac{1}{2}}}}_{(I)} \cdot e^{\underbrace{\left(-\frac{1}{2}(x-\mu)' \cdot C^{-1} \cdot (x-\mu)\right)}_{(II)}}$$

(I): $\dfrac{1}{[illegible]\pi} = \dfrac{1}{2\pi |C|^{\frac{1}{2}}} \Rightarrow |C| = \dfrac{3}{4}, \quad |C^{-1}| = \dfrac{4}{3}$

$-\frac{2}{3}(x_1^2 - x_1 x_2 + x_2^2) = -\frac{1}{2} \; x' \cdot C^{-1} \cdot x$ $\quad$ (da keine konst. Terme $\Rightarrow \mu = 0$)

$$-\frac{2}{3}(x_1^2 - x_1 x_2 + x_2^2) = -\frac{1}{2} (x_1, x_2) \cdot \begin{pmatrix} c_{11}^{1} & c_{21}^{1} \\ c_{12}^{1} & c_{22}^{1} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$(\Leftrightarrow) \quad \frac{4}{3}x_1^2 - \frac{4}{3}x_1 x_2 + \frac{4}{3}x_2^2 = (c_{11}^{1} x_1 + c_{21}^{1} x_2, \; c_{12}^{1} x_1 + c_{22}^{1} x_2) \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \frac{4}{3}x_1^2 - \frac{4}{3}x_1x_2 + \frac{4}{3}x_2^2 = c'_{11}x_1^2 + (c'_{12} + c'_{21})\, x_1 x_2 + c'_{22}\, x_2^2$$

$$\Rightarrow c'_{11} = \frac{4}{3} \qquad c'_{12} + c'_{21} = -\frac{4}{3} \quad , c'_{22} = \frac{4}{3}$$

$$|C^{-1}| = c'_{11} \cdot c'_{22} - c'_{12} \cdot c'_{21} = \frac{16}{9} - c'_{12} \cdot c'_{21} = \frac{4}{3}$$

$$\Rightarrow c'_{12} \cdot c'_{21} = \frac{16}{9} - \frac{4}{3} = \frac{4}{9} \Rightarrow c'_{12} = c'_{21} = -\frac{2}{3}$$

$$C^{-1} = \begin{pmatrix} 4/3 & -2/3 \\ -2/3 & 4/3 \end{pmatrix} \Rightarrow C = \frac{1}{|C^{-1}|} = \begin{pmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{pmatrix}$$

$$= \frac{3}{4} \begin{pmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}$$

b) $$f_{x_1}(x_1) = \frac{1}{\sqrt{2\pi}\;\sigma_1} \cdot e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}} = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x_1^2}{2}}$$

$$f_{x_2}(x_2) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x_2^2}{2}}$$

c) Nicht stoch. unabh. da

$$f_X(x_1, x_2) \neq f_{x_1}(x_1) \cdot f_{x_2}(x_2)$$

TI-ZÜ 14.08.06

Aufgabe 3)

a) $\mu_X(t) = E[X(t)]$

$= E[W_1(t)\cdot\cos(2\pi f_0 t) + W_2(t)\cdot\sin(2\pi f_0 t)]$

$= \underbrace{E[W_1(t)]}_{=0}\cdot\cos(2\pi f_0 t) + \underbrace{E[W_2(t)]}_{=0}\cdot\sin(2\pi f_0 t)$

$= 0$

b) $R_{XX}(t_1, t_2) = E[(W_1(t_1)\cdot\cos(2\pi f_0 t_1) + W_2(t_1)\cdot\sin(2\pi f_0 t_1))$

$\cdot (W_1(t_2)\cdot\cos(2\pi f_0 t_2) + W_2(t_2)\cdot\sin(2\pi f_0 t_2))]$

$= \underbrace{E[W_1(t_1)\cdot W_2(t_2)]}_{R_{W_1W_1}(t)}\cos(2\pi f_0 t_1)\cdot\cos(2\pi f_0 t_2) + \underbrace{E[W_1(t_1)\cdot W_2(t_2)]}_{=E[W_1(t_1)]\cdot E[W_2(t_2)] = 0}\cdot$
$\cos(2\pi f_0 t_1)\cdot\sin(2\pi f_0 t_2)$

$+ \underbrace{E[W_2(t_1)\cdot W_1(t_2)]}_{=0}\cdot\sin(2\pi f_0 t_1)\cdot\cos(2\pi f_0 t_2) + \underbrace{E[W_2(t_1)\cdot W_2(t_2)]}_{R_{W_2W_2}(t)}$
$\cdot\sin(2\pi f_0 t_1)\cdot\sin(2\pi f_0 t_2)$

$t = t_2 - t_1$

$= \delta(t)\cdot\cos(2\pi f_0 t_1)\cdot\cos(2\pi f_0 t_2) + \delta(t)\cdot\sin(2\pi f_0 t_1)\cdot\sin(2\pi f_0 t_2)$

$= \delta(t)\cdot\left(\frac{1}{2}(\cos(2\pi f_0(t_2 - t_1)) + \cos(2\pi f_0(t_2 + t_1))) + \frac{1}{2}(\cos(2\pi f_0(t_2 - t_1)) - \cos(2\pi f_0 \ldots\right.$ $t_2 + t_1$

$= \delta(t)\cdot\cos(2\pi f_0 t)$

$S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(t)\cdot e^{-2\pi i f t}\,dt = 1$

c) $X(t)$ schwach stationär: ✓

Da $\mu_X(t) = 0$ und $R_{XX}(t_1, t_2) = R_{XX}(t)$, $t = t_2 - t_1$

Aufgabe 4)

a) $P(A) = \frac{9}{30}$ $\quad P(B) = \frac{16}{30}$ $\quad P(C) = \frac{5}{30}$

b)

$X \backslash Y$	A	B	C
A	$1/30$	$7/30$	$1/30$
B	$6/30$	$6/30$	$4/30$
C	$2/30$	$3/30$	0

c)

$X \backslash Y$	A	B	C	
A	$1/9$	$7/9$	$1/9$	$\Sigma = 1$
B	$6/16$	$6/16$	$4/16$	$\Sigma = 1$
C	$2/5$	$3/5$	0	$\Sigma = 1$

$P(Y|X) = \frac{P(X,Y)}{P(X)}$

d)

$$H(X) = -\sum_{x \in \{A,B,C\}} p(x) \cdot \log_2 p(x) = 1.44$$

e) $$H(Y|X) = -\sum_{x,y \in \{A,B,C\}} \sum p(x,y) \cdot \log_2 p(x,y) = \ldots = 1.23$$

Aufgabe 5)

a) $$I(X,(Y_1,Y_2)) = H(Y_1,Y_2) - H(Y_1,Y_2 \mid X)$$

$$\leq H(Y_1) + H(Y_2) - \underbrace{H(Y_1,Y_2 \mid X)}_{\leq H(Y_1|X) + H(Y_2|X)}$$

$H(Z_1,Z_2) \leq H(Z_1) + H(Z_2)$

$H(Y_1,Y_2|X)$

$\leq H(Y_1|X) + H(Y_2|X)$

$$\leq H(Y_1) + H(Y_2) - H(Y_1|X) - H(Y_2|X)$$

$$= H(Y_1,X) + H(Y_2,X)$$

$$= H(X,Y_1) + H(X,Y_1)$$

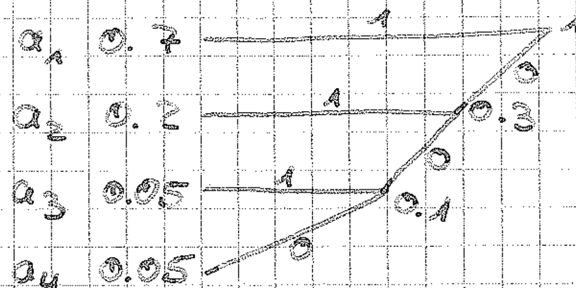
Aufgabe 7)

k	1	2	3	4
$p(x_k)$	0.7	0.2	0.05	0.05

a) $H = -\sum_{k=1}^{4} p(x_k) \cdot \log_2(p(x_k)) = \ldots = 1.256 \frac{\text{bit}}{\text{Symbol}}$

b) $H_{bl} = 2 \cdot H = 2 \cdot 1{,}256 = 2{,}512 \frac{\text{bit}}{\text{Symbolpaar}}$

c)



$a_1 \to 1$

$a_2 \to 01$

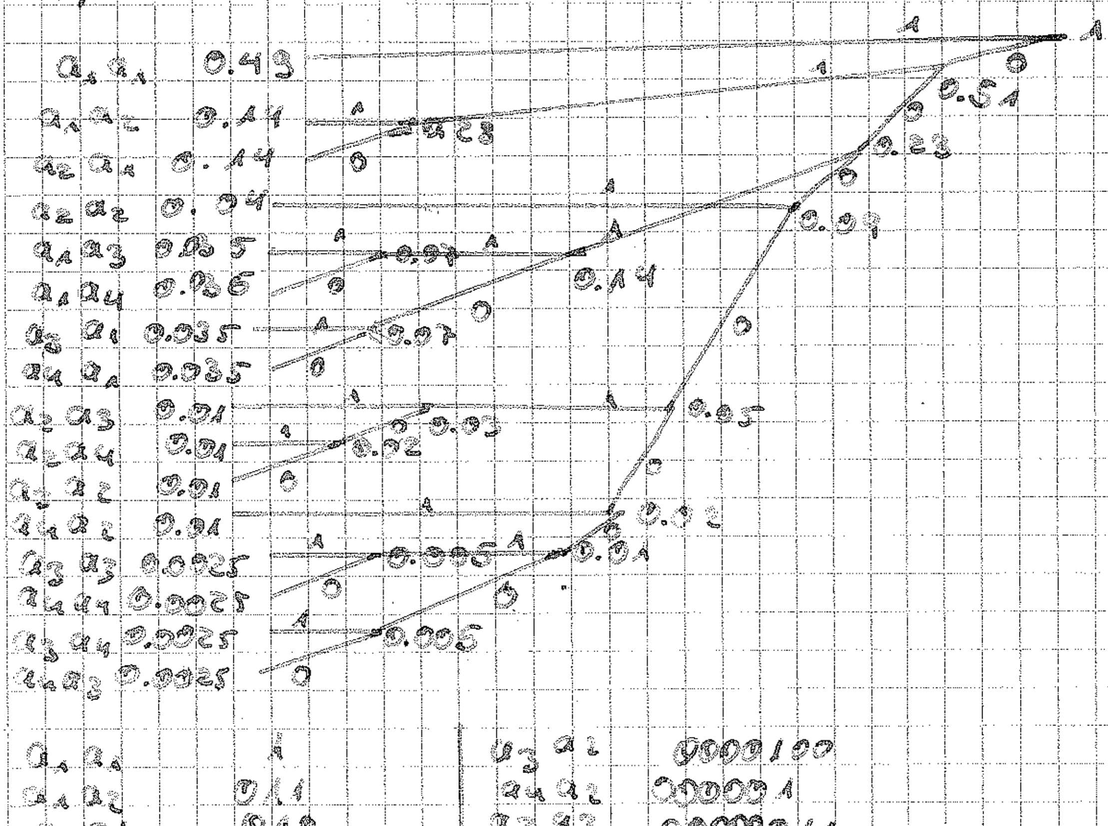
$a_3 \to 001$

$a_4 \to 000$

$\bar{n}_a = \sum_{k=1}^{4} n_k \cdot p(a_k) = 1 \cdot 0.7 + 2 \cdot 0.2 + 3 \cdot 0.05 + 3 \cdot 0.05$

$= 1.4 \frac{\text{bit}}{\text{Symbol}}$

d)



$a_1 a_1$	1	$a_3 a_2$	0000100
$a_1 a_2$	011	$a_4 a_2$	000001
$a_2 a_1$	010	$a_3 a_3$	00000011
$a_2 a_2$	0001	$a_4 a_4$	00000010
$a_1 a_3$	00111	$a_3 a_4$	00000001
$a_1 a_4$	00110	$a_4 a_3$	00000000
$a_3 a_1$	00101		
$a_4 a_1$	00100		
$a_2 a_3$	000011		
$a_2 a_4$	000010		

$\bar{n} = \sum_{i=1}^{4} \sum_{j=1}^{4} n_{ij} \cdot p(a_i a_j) = \ldots = 2.53 \frac{\text{bit}}{\text{Symbolpaar}} = 1.265 \frac{\text{bit}}{\text{Symbol}}$

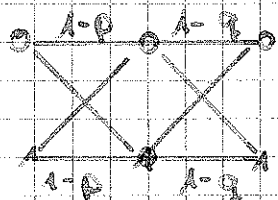
b) $I(X,(Y_1,Y_2)) = H(Y_1,Y_2) - H(Y_1,Y_2|X)$

$= H(Y_1,Y_2) - H(Y_1|X) - H(Y_2|X)$

$\leq H(Y_1) + H(Y_2) - H(Y_1|X) - H(Y_2|X)$

$\Rightarrow I(X,(Y_1,Y_2)) \leq I(X,Y_1) + I(X,Y_2)$

<u>Aufgabe 6)</u>



a) $p_1' = p\cdot(1-p) + (1-p)\cdot q$

$p_2' = p\cdot(1-q) + (1-p)\cdot q$

b) symmetrisch, da $p_1' = p_2' = p'$

c) $C = 0 \Rightarrow p' = \frac{1}{2}$

$p\cdot(1-q) + (1-p)q \overset{!}{=} \frac{1}{2}$

$\Leftrightarrow p - pq + q - pq = p + q - 2pq = p(1-2q) + q = \frac{1}{2}$

$\Leftrightarrow p \overset{q \neq \frac{1}{2}}{=} \frac{\frac{1}{2} - q}{1-2q} = \frac{1}{2}\cdot\frac{2(\frac{1}{2}-q)}{1-2q} = \frac{1}{2}\cdot\frac{(1-2q)}{1-2q} = \frac{1}{2}$

$p \neq \frac{1}{2}$

$\Rightarrow q = \ldots = \frac{1}{2}$

$\Rightarrow C = 0$ für $p = \frac{1}{2}$, q belieb. oder $q = \frac{1}{2}$, p beliebig

TI-ZÜ 14.09.2006

Aufgabe 8)

a) $C = \sum_{i=1}^{3} \left[\log\left(\frac{\nu \lambda_i}{\sigma^2} \right) \right]^+$, $H^* \cdot H = U \cdot \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) \cdot U^*$

$\sum_{i=1}^{3} \left(\nu - \frac{\sigma^2}{\lambda_i} \right)^+ = L = 9$

$$H^* \cdot H = \begin{pmatrix} 20 & 0 & -12 \\ 0 & 8 & 0 \\ -12 & 0 & 20 \end{pmatrix} \longrightarrow EW: 8, 8, 32$$

EV zum EW 8:

$8x = 20x - 12z \Rightarrow -12x = -12z \Rightarrow x = z$

$8y = 8y$

$8z = -12x + 20z$

$\Rightarrow v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}$

EV zum 32

$32x = 20x - 12z \Rightarrow x = -z$

$32y = 8y \Rightarrow y = 0$

$32z = -12x + 20z$

$\Rightarrow v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}$

ν ist Lösung von

$$\Big(\nu - \underbrace{\tfrac{32}{8}}_{4}\Big)^+ + \Big(\nu - \underbrace{\tfrac{32}{8}}_{4}\Big)^+ + \Big(\nu - \underbrace{\tfrac{32}{32}}_{1}\Big)^+ = 9$$

$\Rightarrow \nu = 6$ (Raten)

$$\Rightarrow C = \left[\log\left(\frac{6 \cdot 8}{32}\right)\right]^+ + \left[\log\left(\frac{6 \cdot 8}{32}\right)\right]^+ + \left[\log\left(\frac{6 \cdot 32}{32}\right)\right]^+ = \log\left(\frac{27}{2}\right)$$

b) $X \sim SCN\left(0, U \cdot \operatorname{diag}\left(\nu - \frac{\sigma^2}{\lambda_i}\right)^+ \cdot U^*\right)$

d.h. $X \sim SCN(0, Q)$

$$Q = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 7/2 & 0 & -3/2 \\ 0 & 2 & 0 \\ -3/2 & 0 & 7/2 \end{pmatrix}$$