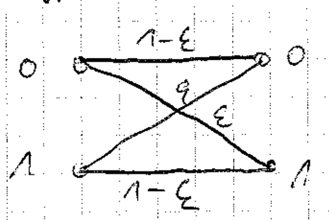
TI

26.02

Zusatzübung

Aufg. 1




a) $P(B_n|A) = \dfrac{P(A|B_n)\cdot P(B_n)}{\sum_i P(A|B_i)\cdot P(B_i)}$

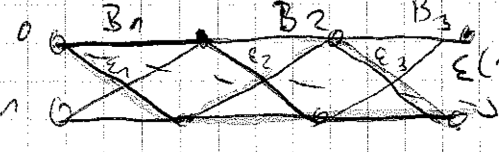
$A_0$: Es wurde eine Null empfangen

$A_1$: Es wurde eine Eins empfangen

$P(B_3|A) = \dfrac{P(A|B_3)\cdot \cancel{P(B_3)}^{1/3}}{\sum_i^3 P(A|B_i)\cdot \cancel{P(B_i)}_{1/3}} = 0{,}3236$

$P(A_1|B) = 0{,}4\,\varepsilon + 0{,}6\,(1-\varepsilon)$

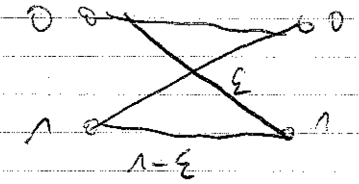
b)




ε (neues ε um einen BSC zu machen)

1 Weg: $(1-\varepsilon_1)(1-\varepsilon_2)\varepsilon_3$
2. Weg: $(1-\varepsilon_1)\varepsilon_2(1-\varepsilon_3)$
3. Weg: $\varepsilon_1\varepsilon_2\varepsilon_3$
4. Weg: $\varepsilon_1(1-\varepsilon_2)(1-\varepsilon_3)$

$\Big\} \sum = \varepsilon = 0{,}284$




$$0{,}4\cdot\varepsilon + 0{,}6(1-\varepsilon) = \underline{\underline{0{,}5432}}$$

c)

$$C = 1 + \varepsilon\cdot\log_2\varepsilon + (1-\varepsilon)\log_2(1-\varepsilon) = 0{,}13915$$

Aufg. 2

a) 1. $f_x(t) \geq 0 \;\forall\, x \in \mathbb{R}$ offenbar erfüllt, da $2-\frac{1}{3}|t| \geq 0$ $\forall\, x \in [-6,6]$

2. $\int_{-\infty}^{\infty} f_x(x) = 1$

$$\int_{-\infty}^{\infty} f_x(t)\,dt = 2\cdot\int_0^6 f_x(x)\,dx = \frac{1}{6}\int_0^6\left(2-\frac{1}{3}x\right)dx$$

$$= \frac{1}{6}\left(2x-\frac{x^2}{6}\right)\Big|_0^6 = 1$$

b)

$$F_x(x) = \int_{-\infty}^{\infty} f_x(t)\,dt$$

$$= \begin{cases} 0 & \text{für } x<-6 \\ \frac{1}{12}\int_{-6}^{x}\left(2+\frac{1}{3}t\right)dt & -6<x<0 \\ \frac{1}{2}+\frac{1}{12}\int_0^6\left(2-\frac{1}{3}t\right)dt & 0<x<6 \\ 1 & 6<x \end{cases} \qquad \begin{cases} 0 & x<-6 \\ \frac{x^2}{72}+\frac{x}{6}+\frac{1}{2} & -6<x<0 \\ -\frac{x^2}{72}+\frac{x}{6}+\frac{1}{2} & 0\leq x\leq 6 \\ 1 & \text{sonst} \end{cases}$$

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c)

$E[X] = 0$ (Produkt aus 2 sym = sym)

$E[(X - E(X))^2] = E[X^2] = \int_{-6}^{6} x^2 \cdot f_x(x)\,dx$

$= 2 \int_{0}^{6} x^2 f_x(x)\,dx = 6$

d) $Y = X + 10$

$X = Y - 10$

$\Rightarrow f_Y(y) = f_X(y - 10)$

e)

$E[Y] = E[X + 10]$

$= E[X] + E[10]$

$= 10$

$E[(Y - E[Y])^2] = E[((X+10) - 10)^2]$

$= E[X^2] = 6$

Aufg. 3

$$H(p) = -\sum_{i=1}^{n} p_i \log_2(p_i) \le \dots$$

Hinweis

$\ln(x) \le x-1 \;\; \forall$ für $x>0$

$$\Leftrightarrow \; -\sum p_i \ln \frac{p_i}{q_i} \le 0$$

$$-\sum p_i \ln \frac{p_i}{q_i} = \sum p_i \ln \frac{q_i}{p_i}$$

$$\le \sum p_i \left(\frac{q_i}{p_i} - 1\right) = \sum q_i - \sum p_i = 0$$

$-D(p\|q)$

$D(p\|q) \ge 0$ wenn beide Vektoren gleiche Zähldichte

Aufg. 4

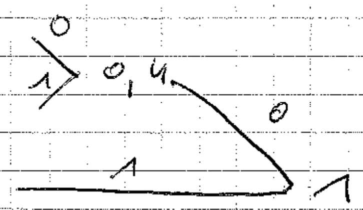
a) $H(X) = -\sum_{i=1}^{3} p_i \log p_i = 1{,}2955$

b) $H(X_1, X_2) = 2 \cdot H(X) = 2{,}5909$

da gedächtnislos

c)

| K | $p_K$ |
|---|---|
| 3 | 0,1 |
| 1 | 0,3 |
| 2 | 0,6 |




| K | $Y_k$ | d | $p_k$ |
|---|---|---|---|
| 1 | 01 | 2 | 0,3 |
| 2 | 1 | 1 | 0,6 |
| 3 | 00 | 2 | 0,1 |

$$\sum_{k=1}^{3} p_k d_k = 1{,}4$$

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d)

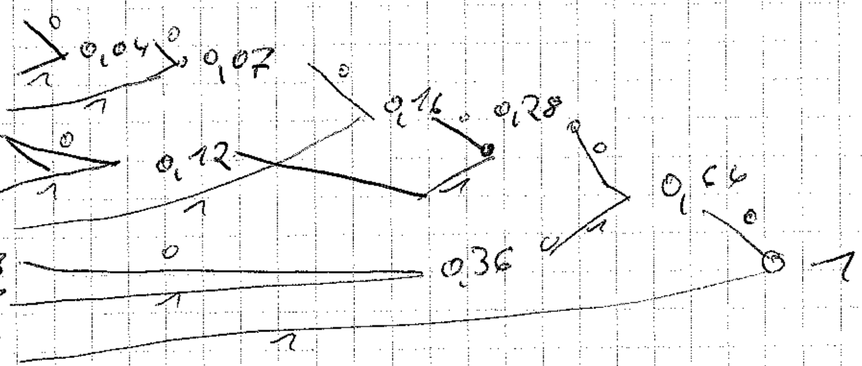
1.)

| Symbol | $P_{KK}$ |
|---|---|
| 11 | 0,09 |
| 12 | 0,18 |
| 13 | 0,03 |
| 21 | 0,18 |
| 22 | 0,36 |
| 23 | 0,06 |
| 31 | 0,03 |
| 32 | 0,06 |
| 33 | 0,01 |

Berechne W'keiten

2. Sortiere nach W'keiten

| | |
|---|---|
| 33 | 0,01 |
| 13 | 0,03 |
| 31 | 0,03 |
| 23 | 0,06 |
| 32 | 0,06 |
| 11 | 0,09 |
| 12 | 0,18 |
| 21 | 0,18 |
| 22 | 0,36 |




3. Codeablesen

| Symbol | Codewort | d |
|---|---|---|
| 11 | 0001 | 4 |
| 12 | 010 | 3 |
| 13 | 000001 | 6 |
| 21 | 011 | 3 |
| 22 | 1 | 1 |
| 23 | 0010 | 4 |
| 31 | 00001 | 5 |
| 32 | 0011 | 3 |
| 33 | 000000 | 6 |

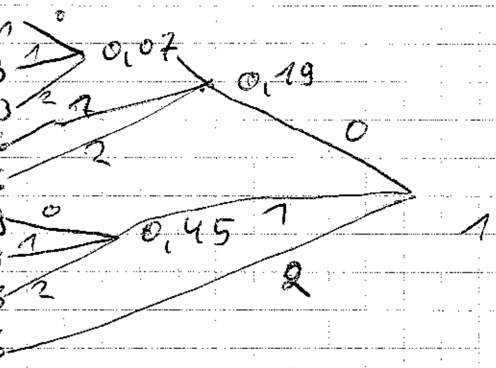
$$\bar{d} = \sum_{KK} P_{KK}\, d_{KK} = 2{,}67$$

$\Rightarrow$ $d_K = 1{,}335$ (einfach durch 2 Teilen)

③

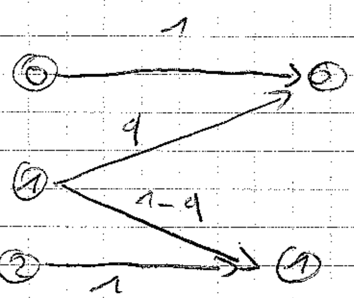
c)

| | K K | |
|---|---|---|
| ~~10~~ | 33 | 0,01 |
| ~~NN~~ | 13 | 0,03 |
| | 31 | 0,003 |
| | 23 | 0,06 |
| | 32 | 0,06 |
| | 11 | 0,09 |
| | 12 | 0,18 |
| | 21 | 0,18 |
| | 22 | 0,36 |




| | |
|---|---|
| 11 | 10 |
| 12 | 11 |
| 13 | 001 |
| 21 | 02 |
| 22 | 2 |
| 23 | 01 |
| 31 | 002 |
| 32 | 02 |
| 33 | 000 |

Aufg 5




a)

$$I(X,Y) = H(X) - H(X|Y)$$
$$= H(Y) - H(Y|X)$$

$$P(y=0) = p_0 \cdot 1 + p_0 \cdot q$$
$$P(y=1) = p_1(1-q) + p_2$$

$\rightarrow H(Y) = -(p_0 + p_1 q)\log(p_0 + q_1 q) - [(p_1(1-q) + p_2)\log(p_1(1-q) + p_2)]$

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$$H(Y|X) = -\sum_{x \in X} \sum_{y \in Y} p(x,y) \cdot \log(p(y|x))$$

$p(0|0) = 1$

$p(0|1) = q$

$p(0|2) = 0$

$p(1|0) = 0$

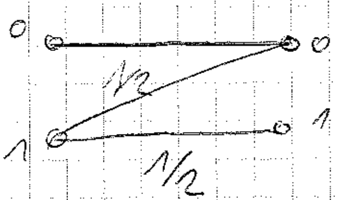
$p(1|1) = 1-q$

$p(1|2) = 1$

Alle Terme mit 0 oder 1 tragen nichts zur Entropie bei!!

$$H(Y|X) = -q\, p_1 \log q - (1-q)\, p_1 \log(1-q)$$

b) $p_2 = 0 \qquad q = 1/2$




$$I(X,Y) = -\log\left(1 - \frac{p_1}{2}\right) + \frac{p_1}{2}\log\left(1 - \frac{p_1}{2}\right) - \frac{p_1}{2}\log\frac{p_1}{2} + p_1 \log \frac{1}{2}$$

$$\frac{d}{dp_1} I(X,Y) = -\frac{1}{1-\frac{p_1}{2}}\left(-\frac{1}{2}\right) + \frac{1}{2}\log\left(1 - \frac{p_1}{2}\right) + \frac{p_1}{2}\,\frac{1}{1-\frac{p_1}{2}}\left(-\frac{1}{2}\right)$$

$$- \frac{1}{2}\log\frac{p_1}{2} - \frac{p_1}{2}\,\frac{2}{p_1}\,\frac{1}{2} + \log\frac{1}{2}$$

$$= \frac{1}{2}\log\frac{1-\frac{p_1}{2}}{p_1/2} + \log\frac{1}{2} \overset{!}{=} 0$$

$$\frac{1}{2}\log\frac{2-\cancel{\frac{p_1}{2}}}{p_1} + \log\frac{1}{2} = 0$$

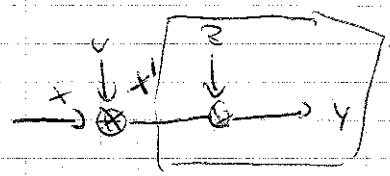
$$\log\frac{2-p_1}{p_1} + \log 1/4 = 0$$

4

$$e^{\log \dots} = e^{0}$$

$$\frac{2-p_1}{p_1} \cdot \frac{1}{4} = 1 \Leftrightarrow 2-p_1 = 4p_1 \Rightarrow p_1 = 2/5$$

Aufg. 6




a) $V = \text{const} = 1/2$

$$x' = 1/2 \cdot x \Rightarrow E[x'^2] = \frac{1}{4} E[x'] = \frac{L}{4}$$

$$C_{kanal} = \frac{1}{2} \log\left(1 + \frac{L'}{\sigma^2}\right)$$

$$\Rightarrow C = 1/2 \log\left(1 + \frac{\frac{L}{4}}{\sigma^2}\right)$$

b) $V = 1$

$$E[Y^2] = E[(X+Z)^2] = E[X^2 + 2XZ + Z^2]$$

$$= E[X^2] + \underbrace{2\,E[XZ]}_{\text{S.U. } \underbrace{E[X]E[Z]}_{0 \text{ da beide } 0}} + E[Z^2]$$

$$= E[X^2] + \sigma^2 \leq P$$

$$E[X^2] \leq P - \sigma^2$$

$$\Rightarrow C = \frac{1}{2} \log\left(1 + \frac{P-\sigma^2}{\sigma^2}\right)$$

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c) $I(x, y|v) = H(x|v) - H(x|y,v)$

$\overset{v \text{ s.u. } x}{=} H(x) - H(x|y,v)$

allg.: $H(R|S) \leq H(R)$

$\geq H(x) - H(x, y)$

$= I(x, y)$

Aufg. 7

a) $C_3 = C_1 - C_2$ bestimme $L_3$

Kapazität $C_1$:

muss immer reell und symmetrisch sein - Immer!

$$H_1^* \cdot H_1 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 5 & 0 & -3 \\ 0 & 4 & 0 \\ -3 & 0 & 5 \end{pmatrix}$$

$$H_1 \cdot H_1^* = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix} \Rightarrow \lambda_1 = 2, \quad \lambda_2 = 4, \quad \lambda_3 = 8$$

$H^+ H$ immer gleiche EW $H H^*$

Waterfilling:

$$\sum_{i=1}^{3} \left( \nu_i - \frac{\sigma_1^2}{\lambda_i} \right)^+ = L_1 \quad , \quad \sigma_1^2 = 16 \quad L_1 = 16$$

$$(\nu_1 - 8)^+ + (\nu_1 - 4)^+ + (\nu_1 - 2)^+ = 16 \Rightarrow \nu_1 = 10$$

$$C = \sum_{i=1}^{3} \left[ \underset{\ln}{\log} \left( \frac{\nu_1 \lambda_i}{\sigma_1^2} \right) \right]^+ = \log\left(\frac{5}{4}\right) + \log\left(\frac{5}{2}\right) + \log 5) \approx 2{,}75$$

Kapazität $C_2$:

$$H_2^* H_2 = \begin{pmatrix} 1-i & 0 & \sqrt{3}+i & 0 \\ 0 & 2+2i & 0 & \sqrt{2}-2i \end{pmatrix} \begin{pmatrix} 1+i & 0 \\ 0 & 2-2i \\ \sqrt{3}-i & 0 \\ 0 & \sqrt{2}+2i \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 14 \end{pmatrix}$$

$\lambda_1 = 6, \ \lambda_2 = 14$, Waterfilling $\sum_{i=1}^{2} \left(\nu_2 - \frac{\sigma_2^2}{\lambda_i}\right)^+ \quad L_2 = 8, \ \sigma_2^2 = 18$

$$\left(\nu_2 - \frac{18}{6}\right)^+ + \left(\nu_2 - \frac{18}{14}\right)^+ = 8 \iff (\nu_2 - 3)^+ + \left(\nu_2 - \frac{9}{7}\right) = 8$$

$$\nu_2 = \frac{43}{7}$$

$$C_2 = \sum_{i=1}^{2} \left[ \log\left(\frac{\nu_2 \cdot \lambda_i}{\sigma_2^2}\right)\right]^+ = \log\left(\frac{43}{21}\right) + \log \frac{43}{9} \overset{\ln}{\approx} 2{,}28$$

$$\Rightarrow C_1 - C_2 = 2{,}75 - 2{,}28 = 0{,}47 = C_3$$

Kapazität $C_3$ $\qquad H_3^* H_3 = (1) = \lambda_1 = 1$

$$\left(\nu_3 - \frac{\sigma_3^2}{1}\right)^+ = L_3 \Rightarrow \nu_3 = L_3 + 18$$

$$\Rightarrow C_3 = \log\left(\frac{\nu_3}{\sigma_3^2}\right) = \log\left(\frac{L_3 + 18}{18}\right) = 0{,}47$$

$$\frac{L_3 + 18}{18} = e^{0{,}47} \Rightarrow L_3 = 18 \cdot e^{0{,}47} - 18 \approx 10{,}75$$

b) Eingabeverteilung $\qquad H_1^* \cdot H_1 = \begin{pmatrix} 5 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 5 \end{pmatrix}$

EV zu $\lambda_1 = 2$:

$$2x = 5x - 3z \Rightarrow x = z \qquad u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$2y = 4y \Rightarrow y = 0 \qquad \text{~~EV zu } \lambda_2 =\text{~~}$$

$$2z = -3x + 5z$$

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EV zu $\lambda_2 = 4$

$u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$4x = 5x - 3z$

$4y = 4y$

$4z = -3x + 5y$

EV zu $\lambda_3 = 8$:

$8x = 5x - 3z \Rightarrow x = -z$

$8y = 4y \Rightarrow y = 0$

$8z = -3x + 5z$

$u_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\Rightarrow X \sim SCN\left(0, \underbrace{u \cdot diag\left(v - \frac{\sigma_1^2}{\lambda_i}\right)^+ u^*}_{Q}\right)$

$$Q = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & & \\ & 6 & \\ & & 8 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 & -3 \\ 0 & 6 & 0 \\ -3 & 0 & 5 \end{pmatrix}$$

Aufg. 8

a) $\max \quad 6x_1 + 15x_2$

s.d. $40x_1 + 26\frac{2}{3}x_2 \leq 4000$ (Leistungsaufnahme)

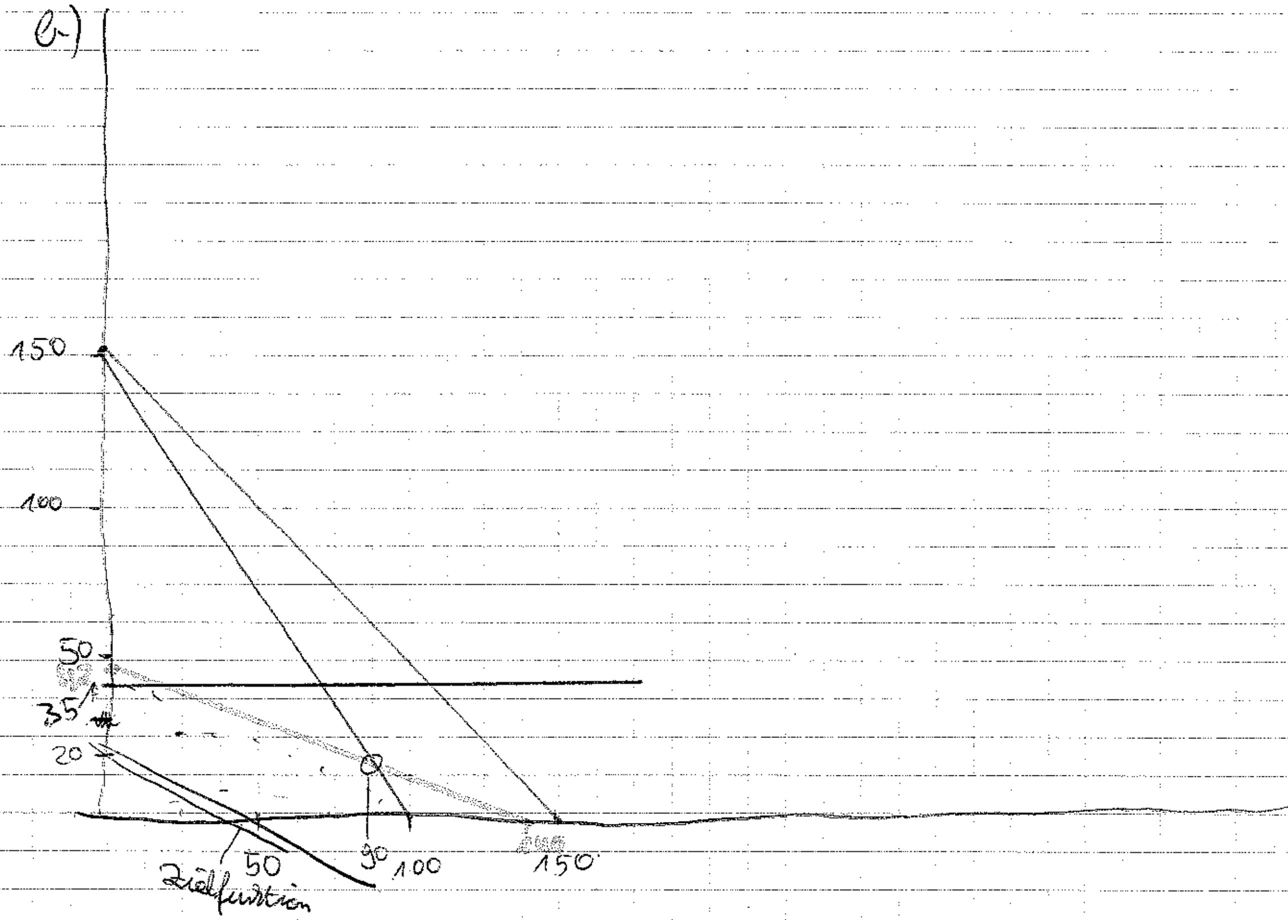
$x_1 + x_2 \leq 150$ (Steckplätze)

$450x_1 + 1500x_2 \leq 63.000$ (Budget)

$x_2 \leq 35$ (Lieferengpass)

$x_1, x_2 \geq 0, \quad x_1, x_2 \in \mathbb{N}_0$

b)




Summe Stellplätze

Summe Leistungsaufnahme

$[x_1, x_2] = [90, 15]$

c) Maximum in Zielfunktion einsetzen in ZF.

$6 \cdot 90 + 15 \cdot 15 = 765$

d) Kosten

$450 \cdot 90 + 1500 \cdot 15 = 63.500$