[ ] Kurztestung 23.2.07

1) 3 E-Mailklassen

$P_W = 0,1$ $P_I = 0,1$ $P_U = 0,8$

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$P_{LW} = 0,1$ $P_{LI} = 0,4$ $P_{LU} = 0,8$

a) L gelöscht

$P_L = P_{LW} \cdot P_W + P_{LI} \cdot P_I + P_{LU} \cdot P_U = \underline{\underline{0,69}}$

b) $P_{b)} = P(W \mid L^c) = \dfrac{P(W, L^c)}{P(L^c)} = \dfrac{P(L^c \mid W) \cdot P_W}{1 - P_L}$

$= \dfrac{(1 - 0,1) \cdot 0,1}{1 - 0,69} = \underline{\underline{0,29}}$

c) $P_{c)} = P(J \mid L) = \dfrac{P(L, J)}{P_L} = \dfrac{P_{LI} \cdot P_I}{P_L} = \underline{\underline{0,058}}$

d) $P_{d)} = P(U \mid L^c) = \dfrac{P(U, L^c)}{P(L^c)} = \dfrac{P(L^c \mid U) \cdot P_U}{P(L^c)} = \dfrac{0,2 \cdot 0,8}{1 - 0,69}$

$= \underline{\underline{0,52}}$

52% ...

A2) $f_X(x) = \begin{cases} \cos^2 \frac{\pi \cdot x}{2} & ; -1 \leq x \leq 1 \\ 0 & ; \text{sonst} \end{cases}$

a) $f_X(x) \geq 0 \quad \forall x \in \mathbb{R}$ erfüllt da $\cos^2$

$\int_{-\infty}^{\infty} f_X(x)\, dx \stackrel{!}{=} 1$

$$\int_{-1}^{1} \cos^2\left(\frac{\pi x}{2}\right) dx \overset{\text{Hinweis}}{=} \int_{-1}^{1} \frac{1}{2}\left(1+\cos(\pi x)\right) dx$$

$$= \int_{-1}^{1} \frac{1}{2}\, dx + \frac{1}{2}\int_{-1}^{1} \cos(\pi x)\, dx$$

$$= \frac{1}{2}\left[1-(-1)\right] + \frac{1}{2}\underbrace{\left[\frac{1}{\pi}\sin(\pi x)\right]_{-1}^{1}}_{=0} = \underline{\underline{1}}$$

b) Verteilungsfunktion

$$F_X(x) = \int_{-\infty}^{x} f_X(s)\, ds$$

$\underline{x \le -1}$ $\quad F_X(x) = \int_{-\infty}^{x} 0\, ds = 0$

$-1 < x < 1$ $\quad F_X(x) = \int_{-1}^{x} \cos^2\left(\frac{\pi s}{2}\right) ds$

$$= \int_{-1}^{x} \frac{1}{2}\left(1+\cos(\pi s)\right) ds$$

$$= \int_{-1}^{x} \frac{1}{2}\, ds + \frac{1}{2}\int_{-1}^{x} \cos(\pi s)\, ds$$

$$= \frac{1}{2}(x+1) + \frac{1}{2\pi}\sin(\pi x)$$

$x \ge 1$ $\quad F_X(x) = \int_{-1}^{1} \cos^2\left(\frac{\pi s}{2}\right) \overset{a)}{=} 1$

$$F_X(x) = \begin{cases} 0 & , x \le -1 \\ \frac{1}{2}(x+1) + \frac{1}{2\pi}\sin(\pi x) & -1 \le x \le 1 \\ 1 & , x \ge 1 \end{cases}$$

c) E-Wert

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \int_{-1}^{1} \frac{1}{2} x \left(1+\cos(\pi x)\right) dx$$

$$= \frac{1}{2}\left[\frac{1}{2}x^2\right]_{-1}^{1} + \frac{1}{2}\left(\left[x\,\frac{1}{\pi}\sin(\pi x)\right]_{-1}^{1} - \int_{-1}^{1}\frac{1}{\pi}\sin \pi x\, dx\right)$$

$= 0 + \frac{1}{2\pi^2} \left[\cos(\pi x) \right]_{-1}^{1} = 0$

d) Varianz

$V(x) = \int_{-\infty}^{\infty} (x - \underbrace{E(x)}_{=0})^2 \, f_x(x)\, dx$

$= \frac{1}{2} \int_{-1}^{1} x^2 (1 + \cos(\pi x))\, dx$

$= \frac{1}{6} [x^3]_{-1}^{1} + \frac{1}{2} \left(\left[x^2 \frac{1}{\pi} \sin(\pi x) \right]_{-1}^{1} - \int_{-1}^{1} 2x \frac{1}{\pi} \sin(\pi x)\, dx \right)$

$= \frac{1}{3} - \frac{1}{\pi} \int_{-1}^{1} x \sin(\pi x)\, dx = \ldots$

$= \frac{1}{3} - \frac{2}{\pi^2} + \frac{1}{\pi^2} \left[\frac{1}{\pi} \sin(\pi x) \right]_{-1}^{1} = \underline{\underline{\frac{1}{3} - \frac{2}{\pi^2}}}$

e) $P(X > 0) = 1 - P(X \leq 0) = 1 - F_x(0) = 1 - \frac{1}{2} = \frac{1}{2}$

Aufg. 3

ZV $X \in \mathbb{N}$

a) $P(X = n) = \left(\frac{1}{2}\right)^{n-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^n$

$\left(P(X \leq n) = 1 - P(X > n) \right)$

b) $P(X \leq n) = \sum_{j=1}^{n} \left(\frac{1}{2}\right)^j = 1 - \frac{1}{2^n}$

c) $E(X) = \sum_{p=1}^{\infty} p \cdot P(X = n) = \sum_{n=1}^{\infty} n \frac{1}{2^n} \overset{\text{Hinweis}}{=} \frac{1/2}{(1 - \frac{1}{2})^2} = \underline{\underline{2}}$

d) $V(X) = \sum_{n=1}^{\infty} (n-2)^2 P(X = n) = \sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^n - 4 \sum_{n=1}^{\infty} n \frac{1}{2^n} + 4 \sum_{n=1}^{\infty} \frac{1}{2^n}$

$\overset{\text{Hinweis}}{=} \frac{\frac{1}{2}(1 + \frac{1}{2})}{(1 - \frac{1}{2})^3} - 4 \cdot 2 + 4 \frac{1/2}{1/2} = \underline{\underline{2}}$

e) $H(X) = -\sum_{n=1}^{\infty} P(X=n) \cdot \log_2 (P(X=n))$

$= -\sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \log_2 \left(\left(\frac{1}{2} \right)^n \right)$

$= \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot n = 2$

f) $g : \mathbb{N} \to \{0,1\}, \quad n \mapsto n \bmod 2$

$H(g(X)) = -\left(P(g(X)=0) \cdot \log_2(\dots) + P(g(X)=1) \cdot \log_2(\dots) \right)$

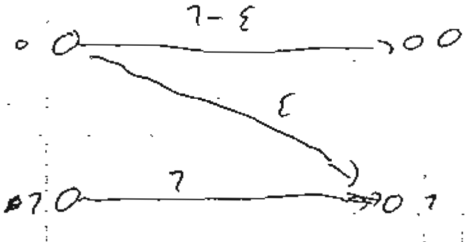
$= -\left(\sum_{n=1}^{\infty} P(X=2n) \cdot \dots \quad \sum P(X=2n-1) \cdot \log_2(\dots) \right)$

$= -\left(\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{2n} \cdot \log_2(\dots) + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{2n-1} \cdot \log_2(\dots) \right)$

$= -\left(\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{n} \cdot \log_2(\dots) + 2 \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{n} \log_2(\dots) \right)$

$= -\left(\frac{1}{3} \cdot \log\left(\frac{1}{3}\right) + \frac{2}{3} \cdot \log\left(\frac{2}{3}\right) \right) = \underline{\underline{0{,}92}}$

A4



$X, Y \in \{0,1\} \quad p_0 = P(X=0), \quad p_1 = P(X=1), \quad \varepsilon \in (0,1)$

a) $I(X,Y) = H(Y) - H(Y|X)$

$H(Y) = -\sum_{y=0}^{1} P(Y=y) \cdot \log_2 (P(Y=y))$

$\overset{\text{Bayes}}{=} - \sum_{j=0}^{1} \sum_{i=0}^{1} P(Y=j \mid X=i) \cdot P(X=i) \cdot$

$\log_2 \left(\sum_{i=0}^{1} P(Y=j \mid X=i) \cdot P(X=i) \right)$

$= - \sum_{j=0}^{1} \Big(P(Y=j \mid X=0) \cdot P(X=0) + P(Y=j \mid X=1) \cdot P(X=1)$

$\cdot \log_2(\dots) \Big)$

$= -[(1-\varepsilon) p_0 + 0) \log_2 ((1-\varepsilon) p_0) + (\varepsilon p_0 + 1 p_1) \log_2 (\varepsilon p_0 + p_1$

$= a - \Big[(1-\varepsilon) p_0 \cdot \log_2((1-\varepsilon) p_0) + (\varepsilon p_0 + (1-p_0)) \log_2 (\varepsilon p_0 + (1-p_0$

$H(Y \mid X) = - \sum_{i=0}^{1} \sum_{j=0}^{1} \underbrace{P(X=i, Y=j)}_{P(Y=j \mid X=i) \cdot P(X=i)} \cdot \log_2 (P(Y=j \mid X=i))$

$= - [P(Y=0 \mid X=0) P(X=0) \cdot \log(P(Y=0 \mid X=0)$

$+ \dots \qquad]$ [4 Fälle]

$= - ((1-\varepsilon) p_0 \log_2 (1-\varepsilon) + \varepsilon p_0 \log_2 (\varepsilon))$

$I(X,Y) = H(Y) - H(Y \mid X) = \dots =$

$= \varepsilon p_0 \log_2 \varepsilon - (1-\varepsilon) p_0 \log_2 p_0 - (1 - p_0 (1-\varepsilon)) \cdot \log(1-p_0(1-$

b) $C = \max_{p_0 p_1} I(X,Y)$

$$\frac{d}{dp_0} I(X,Y) = \frac{1}{\ln 2}\Big(\varepsilon \ln \varepsilon - (1-\varepsilon)(\ln p_0 + 1)$$

$$- \big(-(1-\varepsilon) \ln (1 - p_0(1-\varepsilon))$$

$$+ (1 - p_0(1-\varepsilon)) \cdot \frac{1}{1-p_0(1-\varepsilon)} (-(1-\varepsilon))\big)\Big)$$

$$= \frac{1}{\ln 2}\left(\varepsilon \ln \varepsilon + (1-\varepsilon) \ln \left(\frac{1-p_0(1-\varepsilon)}{p_0}\right)\right) \overset{!}{=} 0$$

$$\Leftrightarrow \quad \ln\left(\frac{1-p_0(1-\varepsilon)}{p_0}\right) = -\frac{\varepsilon \ln \varepsilon}{1-\varepsilon}$$

$$\Leftrightarrow \quad \frac{1-p_0(1-\varepsilon)}{p_0} = \exp\left(-\frac{\varepsilon \ln \varepsilon}{1-\varepsilon}\right)$$

$$\Leftrightarrow \quad \frac{1}{p_0} - (1-\varepsilon) = \exp\left(-\frac{\varepsilon \ln \varepsilon}{1-\varepsilon}\right)$$

$$p_0 = \left((1-\varepsilon) + \exp\left(-\frac{\varepsilon \ln \varepsilon}{1-\varepsilon}\right)\right)^{-1}$$

$$p_1 = 1 - p_0$$

c) $\varepsilon = \frac{1}{2}$

$p_0 = 2/5$

$$C = \frac{1}{2} \cdot \frac{2}{5} \log_2\left(\frac{1}{2}\right) - \frac{1}{2} \cdot \frac{2}{5} \log_2\left(\frac{2}{5}\right) - \left(1 - \frac{2}{5}\left(1-\frac{1}{2}\right)\right) \cdot \log_2\left(1 - \frac{2}{5}\left(1-\frac{1}{2}\right)\right)$$

$$= \underline{\underline{0{,}322}}$$

Aufgabe 5

a) mögliche Ausgabewerte $\{0,1\}^3$

Ausgabewert b	W'keit $p(b)$		
$(0,0,0)$	$\frac{1}{4}(1-\varepsilon)^3$	$+\ \frac{3}{4}\,\varepsilon^3$	$= 11/108$
$(0,0,1)$	$\frac{1}{4}(1-\varepsilon)^2\cdot\varepsilon$	$+\ \frac{3}{4}\,\varepsilon^2(1-\varepsilon)$	$= 10/108$
$(0,1,0)$		"	
$(1,0,0)$		"	
$(0,1,1)$	$\frac{1}{4}(1-\varepsilon)\cdot\varepsilon^2$	$+\ \frac{3}{4}\,\varepsilon\cdot(1-\varepsilon)^2$	$= 14/108$
$(1,0,1)$		"	
$(1,1,0)$		"	
$(1,1,1)$	$\frac{1}{4}\,\varepsilon^3$	$+\ \frac{3}{4}(1-\varepsilon)^3$	$= 25/108$

b) $p_R = \frac{1}{4}(1-\varepsilon)^3 + \frac{3}{4}(1-\varepsilon)^3 = \frac{8}{27}$

c) $p_E = \frac{1}{4}\varepsilon^3 + \frac{3}{4}\varepsilon^3 = \frac{1}{27}$

d) ML-Detektion $p(b\,|\,c)$

b \ c	$(0,0,0)$		$(1,1,1)$	
$(0,0,0)$	$8/27$	×	$1/27$	
$(0,0,1)$	$4/27$	×	$2/27$	
$(0,1,0)$	$4/27$	×	$2/27$	
$(1,0,0)$	$4/27$	×	$2/27$	
$(0,1,1)$	$2/27$		$4/27$	×
$(1,0,1)$	$2/27$		$4/27$	×
$(1,1,0)$	$2/27$		$4/27$	×
$(1,1,1)$	$1/27$		$8/27$	×

$h_{ML}(0,0,0) = (0,0,0)$

e) ME-Dekodierung, $P(c \mid b) = \frac{P(b \mid c) \cdot P(c)}{P(b)}$

b \ c	0,0,0	1,1,1	
0 0 0	0,73 X	0,27	
0 0 1	0,4	0,6	X
0 1 0	0,4	0,6	X
1 0 0	0,4	0,6	X
0 1 1	0,14	0,86	X
1 0 1	0,14	0,86	X
1 1 0	0,14	0,86	X
1 1 1	0,04	0,96	X

$g_{ME} \quad (0,0,0) = (0,0,0)$

$g_{ME} \quad (0,0,1) = (1,1,1)$

$\vdots \quad \vdots = \quad "$

Aufg. 6

a) $H_1^* \cdot H_1 = \begin{pmatrix} 1 & 0 & \sqrt{3} & 0 \\ 0 & 2 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$

EW von $H_1^* \cdot H_1 = 4, 6 = \lambda_1, \lambda_2$

$\sum_{i=1}^{2} \left(\vartheta - \frac{\sigma^2}{\lambda_i} \right)^+ = L$ $\quad [(\ )^+$: positiver Anteil $]$

$= \left(\vartheta - \frac{8}{4} \right)^+ + \left(\vartheta - \frac{8}{6} \right)^+ = 4$

$\Rightarrow \left(\vartheta - 2 \right)^+ + \left(\vartheta - \frac{4}{3} \right) = 4 \Rightarrow \vartheta = \frac{11}{3}$

$C_1 = \sum_{i=1}^{2} \left(\ln \left(\frac{\vartheta \lambda_i}{\sigma^2} \right) \right)^+$

$= \ln \left(\frac{11}{6} \right) + \ln \left(\frac{11}{4} \right) = \ln \left(\frac{121}{24} \right)$

b) $X \sim SCN\left(\underline{0}, U \cdot \mathrm{diag}\left(\tilde{\nu} - \frac{\sigma^2}{\lambda_i}\right)^+ \cdot U^*\right)$

In diesem Fall:

$X \sim SCN(\underline{0}, \underline{Q})$

mit $Q = \begin{pmatrix} 5/3 & 0 \\ 0 & 7/3 \end{pmatrix} = U \mathrm{diag}(\ldots) U^*$

c) $H_2 = 1 \quad L = 4$

SISO-Kanal besitzt die Kapazität

$C_2 = \ln\left(\frac{\nu' \cdot 1}{\tau^2}\right)$

$\left(\nu' - \frac{\tau^2}{1}\right)^+ = 4$

$\Rightarrow \nu' = \tau^2 + 4$

$\Rightarrow C_2 = \ln\left(\frac{\tau^2+4}{\tau^2}\right) = \ln\left(1 + \frac{4}{\tau^2}\right) \overset{!}{=} \ln\left(\frac{121}{24}\right) = C_1$

$\Rightarrow 1 + \frac{4}{\tau^2} = \frac{121}{24} \Rightarrow \tau^2 = \frac{4}{\frac{121}{24} - 1} \approx 0{,}99$

Aufg. 7

$$P_n = \{ \underline{p} = (p_1, \dots, p_n) \mid p_i \geq 0 \ \forall i, \ \sum_{i=1}^{n} p_i = 1 \}$$

Sei $q, r \in P_n$

Zu zeigen: $\alpha \cdot q + (1-\alpha) \cdot r \in P_n$

~~und~~ mit $\alpha \in [0,1]$

1. $\alpha \cdot q_i + (1-\alpha) r_i \overset{!}{\geq} 0$ für $i = 1, \dots, n$

korrekt, da α, $(1-\alpha)$, q_i und r_i ≥ 0

2. $\sum_{i=1}^{n} (\alpha q_i + (1-\alpha) r_i) \overset{!}{=} 1$

$\sum_{i=1}^{n} (\alpha q_i + (1-\alpha) r_i) =$

$\alpha \underbrace{\sum_{i=1}^{n} q_i}_{1 \text{ da } q \in P} + (1-\alpha) \underbrace{\sum_{i=1}^{n} r_i}_{1 \text{ da } r \in P}$

$= \alpha + (1-\alpha) = 1$

Aufg 8

a) $\max \quad 15x_1 + 25x_2$

s.t.

$x_1 + 2x_2 \leq 85$ (Signalisierungskanal)

$x_1 + x_2 \leq 75$ (Netzkanäle)

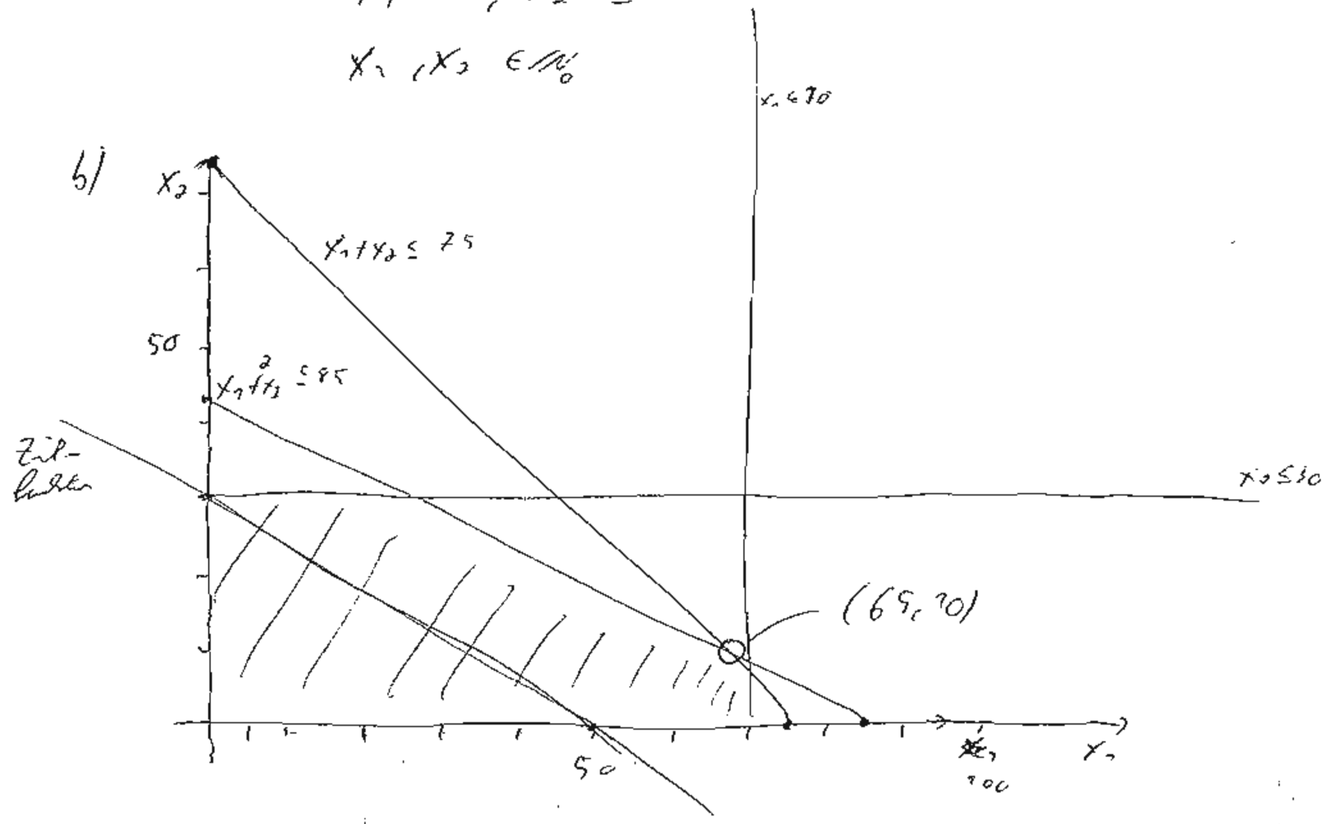
$x_2 \leq 30$ (max. Nutzerzahl)

$x_1 \leq 70$ (")

$x_1 \geq 0,\ x_2 \geq 0$

$x_1, x_2 \in \mathbb{N}_0$

b)



opt. Lsg. für $-\frac{x_1}{2} + \frac{85}{2} = -x_1 + 75$

$\Leftrightarrow x_1 = 65$

$x_2 = -x_1 + 75 = 10$

c) Maximum $15 \cdot 65 + 10 \cdot 25 = 1225$

wird angenommen bei $(65 | 10)$