31. Aufgabe

$$g(x) = \int_2^{x+x^3} \frac{1+t+\sqrt{t^4-1}}{1+t^2\sqrt{t^2+1}}\,dt \quad , \quad D(g) := [1,\infty)$$

zu berechnen: $g'(x)$

Setze $F(y) := \int_2^y \frac{1+t+\sqrt{t^4-1}}{1+t^2\sqrt{t^2+1}}\,dt$ und $h(x) := x+x^3$

mit $D(h) := [1,\infty)$ $\left(\Rightarrow W(h) := [2,\infty)\right)$ und $D(F) = [2,\infty)$

Dann gilt $g(x) = F(h(x))$

Da $f(t) := \frac{1+t+\sqrt{t^4-1}}{1+t^2\sqrt{t^2+1}}$ stetig auf $(2,\infty)$ folgt nach dem

1. HDI (Satz VII.1.1): $F'(y) = f(y)$ in $(2,\infty)$.

Es folgt:

$$g'(x) = F'(h(x)) \cdot h'(x) \quad ; \text{ Kettenregel}$$

$$= f(h(x)) \cdot (1+3x^2)$$

$$= \underline{\underline{\frac{1+(x+x^3)+\sqrt{(x+x^3)^4-1}}{1+(x+x^3)^2 \cdot \sqrt{(x+x^3)^2+1}}(1+3x^2)}} \quad \text{in } (1,\infty)$$

zu zeigen: g ist streng monoton wachsend in $(1,\infty)$

Es gilt für $x \in (1,\infty)$:

$$g'(x) = \frac{1+x+x^3+\sqrt{(x+x^3)^4-1}}{1+(x+x^3)^2\sqrt{(x+x^3)^2+1}} \cdot (1+3x^2)$$

$$\geq \frac{1+1+1+\sqrt{(1+1)^4-1}}{1+(x+x^3)^2\sqrt{(x+x^3)^2+1}} (1+3\cdot 1^2) > 0$$

Somit folgt, dass g in $(1,\infty)$ streng monoton wachsend ist

zu zeigen: g ist glm. stetig in $(1,\infty)$

Es gilt für $x \in (1,\infty)$:

$$|g'(x)| = \frac{1+x+x^3+\sqrt{(x+x^3)^4-1}}{1+(x+x^3)^2\sqrt{(x+x^3)^2+1}} (1+3x^2)$$

$$\leq \frac{x^6+x^6+x^6+\sqrt{(x^3+x^3)^4}}{(x^3)^2\sqrt{(x^3)^2}} \cdot (x^2+3x^2)$$

$$= \frac{3x^6+(2x^3)^2}{x^6\cdot x^3} \cdot 4x^2$$

$$= \frac{28\cdot x^8}{x^9} = \frac{28}{x^9} \leq 1$$

Es folgt (nach TEIL A-Aufgabe 20), dass g in $(1,\infty)$ glm. stetig ist.

2. Aufgabe

a) zu berechnen: $\int_0^1 x^3 e^x \, dx$:

Setze $f(x) = x^3$ und $g(x) = e^x$. Dann ist

$$\int_0^1 x^3 e^x \, dx = \int_0^1 f(x) g(x) \, dx = \left[f(x) \, G(x)\right]_0^1 - \int_0^1 f'(x) \, G(x) \, dx$$

$$= \left[x^3 \cdot e^x\right]_0^1 - \int_0^1 3x^2 e^x \, dx$$

$$= e^1 - \left(\left[3x^2 e^x\right]_0^1 - \int_0^1 6x e^x \, dx \right)$$

$$= e^1 - \left(3e^1 - \left(\left[6x e^x\right]_0^1 - \int_0^1 6 \cdot e^x \, dx \right) \right)$$

$$= e^1 - \left(3e^1 - \left(6e^1 - \left[6 \cdot e^x\right]_0^1 \right) \right)$$

$$= e^1 - \left(3e^1 - 6e^1 + 6e^1 - 6 \cdot e^0 \right) = \underline{\underline{6 - 2e}}$$

b) zu berechnen: $\int_1^2 x^2 \log(x) \, dx$:

$$\int_1^2 x^2 \log(x) \, dx = \left[\log(x) \left(\tfrac{1}{3} x^3\right)\right]_1^2 - \int_1^2 \frac{1}{x} \cdot \left(\tfrac{1}{3} x^3\right) dx$$

$$= \log(2) \frac{8}{3} - \int_1^2 \frac{1}{3} x^2 \, dx = \underline{\underline{\log(2) \cdot \frac{8}{3} - \frac{7}{9}}}$$

c) zu berechnen: $\int_2^3 \log\left(1 + \frac{1}{x}\right) dx$:

$$\int_2^3 \log\left(1 + \tfrac{1}{x}\right) dx = \int_2^3 \log\left(1 + \tfrac{1}{x}\right) \cdot 1 \, dx$$

$$= \left[\log\left(1 + \tfrac{1}{x}\right) \cdot x\right]_2^3 - \int_2^3 \frac{1}{1 + 1/x} \cdot \left(-\frac{1}{x^2}\right) \cdot x \, dx$$

$$= 3\cdot\log(1+\tfrac{1}{3}) - 2\cdot\log(1+\tfrac{1}{2}) + \int_2^3 \frac{1}{x+1}\,dx$$

$$= 3\cdot\log(1+\tfrac{1}{3}) - 2\log(1+\tfrac{1}{2}) + \Big[\log(1+x)\Big]_2^3$$

$$= 3\log(1+\tfrac{1}{3}) - 2\log(1+\tfrac{1}{2}) + \log(4) - \log(2) = \ldots = 10\log(2) - 6\log($$

33. Aufgabe

a) $$\int_{\log(2)}^{\log(3)} \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int_{\log(2)}^{\log(3)} \frac{f'(x)}{f(x)}\,dx \quad ; \text{ mit } f(x) = e^x + e^{-x}$$

$$= \int_{\log(2)}^{\log(3)} g(f(x))\cdot f'(x)\,dx \; ; \text{ mit } g(y) = \frac{1}{y}$$

Satz VII.2.2 $$\overset{\downarrow}{=} \int_{f(\log(2))}^{f(\log(3))} g(y)\,dy$$

$$= \Big[G(y)\Big]_{f(\log(2))}^{f(\log(3))} = \Big[\log(y)\Big]_{5/2}^{10/3} = \log(\tfrac{10}{3}) - \log(\tfrac{5}{2})$$

b) $$\int_0^2 \frac{x^7}{x^4+2}\,dx = \int_0^2 \frac{x^7}{\psi(x)}\,dx \qquad ; \text{ mit } \psi(x) = x^4 + 2$$

$$= \int_0^2 \frac{\frac{1}{4}x^4}{\psi(x)}\psi'(x)\,dx \qquad ; \psi'(x) = 4x^3$$

$$= \int_{\psi(0)}^{\psi(2)} \frac{\frac{1}{4}\varphi(y)^4}{\psi(\varphi(y))}\,dy \qquad ; \varphi(y) = \psi^{-1}(y) = \sqrt[4]{y-2}$$

Kor. zu Satz VII.2.2 $$= \int_2^{18} \frac{\frac{1}{4}(y-2)}{y}\,dy = \frac{1}{4}\left(\int_2^{18} 1 - \frac{2}{y}\,dy\right) = \frac{1}{4}\Big(16 - 2\cdot(\log(18) - \log(2))\Big)$$

c) $\int_0^1 x^7 \sqrt{4x^8+1}\, dx = \int_1^5 \sqrt{y} \cdot \frac{1}{32}\, dy \qquad ; \; y = 4x^8+1$

$$\Downarrow$$

$dy = 32x^7\, dx$

$$= \frac{1}{32}\left[\frac{2}{3} y^{3/2}\right]_1^5$$

$$= \underline{\underline{\frac{2}{96}\left(5^{3/2}-1\right)}}$$

d) $\int_{\log(2)}^{\log(3)} \frac{1}{(e^x+e^{-x})^2}\, dx = \int_2^3 \frac{1}{(y+\frac{1}{y})^2} \cdot \frac{1}{y} \cdot dy \qquad ; \; y = e^x$

$$\Downarrow$$

$dy = e^x dx = y\, dx$

$$= \int_2^3 \frac{y}{(y^2+1)^2}\, dy \qquad ; \quad z = y^2+1 \Rightarrow dz = 2y\, dy$$

$$= \int_5^{10} \frac{1}{z^2} \cdot \frac{1}{2}\, dz = \frac{1}{2} \cdot \left[-\frac{1}{z}\right]_5^{10} = \underline{\underline{\frac{1}{2}\left(\frac{1}{5}-\frac{1}{10}\right)}} = \underline{\underline{\frac{1}{20}}}$$

e) $\int_0^{\frac{1}{3}\sqrt{3}} \frac{x^3}{\sqrt{1-x^2}}\, dx = \int_0^{\pi/3} \sin^3(y)\, dy \qquad ; \; y = \arcsin(x) \Rightarrow x = \sin(y)$

$$\Downarrow$$

$dy = \frac{1}{\sqrt{1-x^2}}\, dx$

$$= \int_0^{\pi/3} \sin(y) \cdot (1-\cos^2(y))\, dy \qquad ; \; \sin^2(y)+\cos^2(y) = 1$$

$$= \int_1^{1/2} -(1-z^2)\, dz \qquad ; \; z = \cos(y) \Rightarrow dz = -\sin(y)\, dy$$

$$= \int_{1/2}^1 (1-z^2)\, dz = \frac{1}{2} - \left[\frac{1}{3} z^3\right]_{1/2}^1 = \frac{1}{2} - \left(\frac{1}{3} - \frac{1}{3} \cdot \frac{1}{8}\right)$$

$$= \underline{\frac{5}{24}}$$

7) $\int_1^2 \frac{1}{\sqrt{x(4-x)}}\,dx = \int_1^2 \frac{1}{\sqrt{4x-x^2}}\,dx$

$= \int_1^2 \frac{1}{\sqrt{-(x-2)^2+4}}\,dx$

$= \int_{-1}^0 \frac{1}{\sqrt{4-y^2}}\,dy \qquad ; \; y = x-2 \Rightarrow dy = dx$

$= \int_{-1}^0 \frac{1}{2}\cdot\frac{1}{\sqrt{1-(y/2)^2}}\,dy$

$= \int_{-1/2}^0 \frac{1}{\sqrt{1-z^2}}\,dz \qquad ; \; z = \frac{y}{2} \Rightarrow dz = \frac{1}{2}dy$

$= \left[\arcsin(z)\right]_{-1/2}^{0} = 0-\left(-\frac{\pi}{6}\right) = \underline{\underline{\frac{\pi}{6}}}$

4. Aufgabe

$$\int \frac{3}{3\cos(x)-5}\,dx \qquad ; \quad y = \tan\left(\tfrac{x}{2}\right) \Rightarrow dy = \frac{1}{\cos^2\left(\frac{x}{2}\right)} \cdot \frac{1}{2}\,dx$$

$$\Downarrow$$

$$\cos(x) = \frac{1-y^2}{1+y^2}$$

$$= \int \frac{3}{3\cdot\left(\frac{1-y^2}{1+y^2}\right)-5}\; 2\cos^2\left(\tfrac{x}{2}\right)\,dy$$

$$= \int \frac{3}{3\left(\frac{1-y^2}{1+y^2}\right)-5}\,\big(1+\cos(x)\big)\,dy \qquad ; \quad 2\cos^2(A) = 1+\cos(2A)$$

$$= \int \frac{3}{3\left(\frac{1-y^2}{1+y^2}\right)-5}\left(1+\frac{1-y^2}{1+y^2}\right)dy$$

$$= \int \frac{3(1+y^2)}{3(1-y^2)-5(1+y^2)}\left(\frac{1+y^2}{1+y^2}+\frac{1-y^2}{1+y^2}\right)dy$$

$$= \int \frac{3}{-2-8y^2}\cdot 2\,dy$$

$$= -6\int \frac{1}{2+8y^2}\,dy$$

$$= -3\int \frac{1}{1+(2y)^2}\,dy \quad = -\frac{3}{2}\int \frac{1}{1+z^2}\,dz \qquad ; \quad z = 2y \Rightarrow dz = 2\,dy$$

$$= -\frac{3}{2}\arctan(z)+C = \underline{\underline{-\frac{3}{2}\arctan\left(2\tan\left(\tfrac{x}{2}\right)\right)+C}}\,, \; C\in\mathbb{R}$$