B 35 ① HM UG ⑦ 08.06.09

a) $= \int \frac{\cos^2(x) + \sin^2(x)\,dx}{\sin^2(x) + 4\cos^2(x)} = \frac{1}{4}\int \frac{1+\tan^2(x)}{\frac{1}{4}\tan^2(x)+1}\,dx$

Subst.
$u = \frac{1}{2}\tan(x)$
$du = \frac{1}{2}(1+\tan^2(x))\,dx$

$\overset{\text{Subst.}}{=} \frac{1}{2}\arctan\left(\frac{1}{2}\tan(x)\right) + C, \; C \in \mathbb{R}$

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b) $\int \frac{1}{\sin(x)}\,dx = \ln\left|\tan\left(\frac{x}{2}\right)\right| + C, \; C \in \mathbb{R}$

B36 $\int \frac{x^6\,dx}{x^3-5x^2+7x-3} = \frac{1}{4}x^4 + \frac{5}{3}x^3 + 9x^2 + 58x + \frac{792}{4}\cdot\ln|x-3|$

$- \frac{13}{4}\ln|x-1| + \frac{1}{2}\frac{1}{x-1} + \begin{cases} C_1, & x \in (-\infty, 1) \\ C_2, & x \in (1,3) \\ C_3, & x \in (3,\infty) \end{cases}$

B 37 analog zu A 37

zeige: $-\frac{x^3}{2} \le f(x) - \frac{1}{2}x^2 \le 0 \quad \forall x \in (0,\pi)$

$g(t) = \arctan(\sin(t))\,dt$ entwickle $g(t)$ um $t_0 = 0$

$g(0) = 0 \quad g'(t) = \frac{\cos(t)}{1+\sin^2(t)} \Rightarrow g'(0) = 1$

$g''(t) = \frac{-\sin(t)(1+\sin^2(t)) - \cos(t)(2\sin(t)\cdot\cos(t))}{(1+\sin^2(t))^2} =$

$\Rightarrow g(t) = 0 + \frac{1}{1!}(t-0) + \frac{1}{2!}(t-0)^2 \cdot g''(\xi) \qquad \xi \in (0,t)$

$= t + \frac{t^2}{2}\cdot g''(\xi)$ für ein $\xi \in (0,\pi)$ gilt:

(i) $\sin(\xi) > 0, \; \cos^2(\xi) \ge 0 \Rightarrow g''(\xi) \le 0$

(ii) $\sin(\xi), \cos^2(\xi) \le 1$

$-3 \le g''(\xi) \le 0$

$g''(t) = \frac{-\sin(t)}{1+\sin^2(t)} - \frac{2\sin(t)\cos^2(t)}{(1+\sin^2(t))^2}$

$\rightarrow -1 - 2 = -3$

②

$$\arctan(\sin(t)) = t + \frac{t^2}{2} g''(\xi) \leq t$$

$$\arctan(\sin(t)) = t + \frac{t^2}{2} g''(\xi) \geq t - \frac{3}{2} t^2$$

Damit folgt für alle $x \in (0,\pi)$

(i) $f(x) = \int_0^x \arctan(\sin(t))\, dt \leq \int_0^x t\, dt = \frac{1}{2} t^2 \Big|_0^x = \frac{x^2}{2}$

$\Leftrightarrow f(x) - \frac{x^2}{2} \leq 0$

(ii) $f(x) = \int_0^x \arctan(\sin(t))\, dt \geq \int_0^x t - \frac{3}{2} t^2\, dt = \frac{1}{2} x^2 - \frac{x^3}{2}$

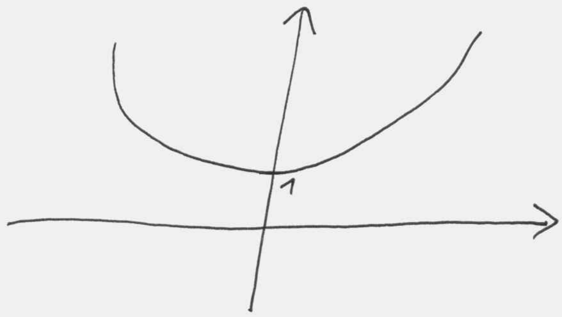
$\Leftrightarrow f(x) - \frac{x^2}{2} \geq -\frac{x^3}{2}$

insgesamt: $-\frac{x^3}{2} \leq f(x) - \frac{x^2}{2} \leq 0$

B 38

$$\frac{\cosh(\xi)\, x^{2n+2}}{(2n+2)!}$$

$$\leq \cosh(x) \cdot \frac{x^{2n+2}}{(2n+2)!} \longrightarrow 0$$



$\cosh(x) > \cosh(\xi)$

(ξ) zwischen 0 und x

B 39

c) $= \lim\limits_{x \to 0_+} \left(\dfrac{\ln(1+x) - x}{x \cdot \ln(1+x)} \right)$

$f(x) := \ln(1+x) - x \xrightarrow{x \to 0_+} 0$

$g(x) := x \cdot \ln(1+x) \longrightarrow 0$

Form „$\frac{0}{0}$" sieht nach (-> L'Hospital) aus...

$$f'(x) = \frac{1}{1+x} - 1 = \frac{-x}{1+x}$$

$$g'(x) = \ln(1+x) + \frac{x}{1+x}$$

$$\frac{f'(x)}{g'(x)} = \frac{-x}{(1+x)\ln(1+x) + x}$$

$u(x) = -x$

$v(x) = (1+x)\ln(1+x) + x$

$\frac{u}{v} = \frac{0}{0} \Rightarrow u' = -1$

$v' = \ln(1+x) + \frac{1+x}{1+x} + 1$

$= \ln(1+x) + 2$

$$\frac{u'}{v'} = \frac{-1}{\ln(1+x) + 2} \xrightarrow{x \to 0} -\frac{1}{2}$$

Regel von L'Hospital liefert:

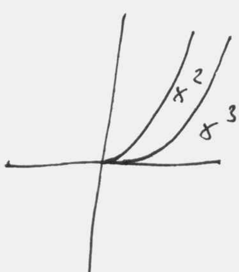
$$\lim_{x \to 0_+} \frac{f(x)}{g(x)} = -\frac{1}{2}$$

mit $\mathcal{O}$-Kalkül

$$\lim_{x \to 0_+} \frac{\ln(1+x) - x}{x \ln(1+x)}$$

$$\ln(1+x) \underset{x_0 = 0}{=} 0 + \frac{1}{1!}x + \frac{-1}{2!}x^2 + \mathcal{O}(x^3) = x - \frac{x^2}{2} + \mathcal{O}(x^3)$$

$$\lim_{x \to 0_+} \frac{\cancel{x} - \frac{x^2}{2} + \mathcal{O}(x^3) - \cancel{x}}{x\left(x - \frac{x^2}{2} + \mathcal{O}(x^3)\right)} = \lim_{x \to 0_+} \frac{-\frac{x^2}{2} + \mathcal{O}(x^3)}{x^2 + \mathcal{O}(x^3)}$$



+ alles was größer ist ($x^4, \ldots, x^n$)

$$= \lim_{x \to 0_+} \frac{-\frac{1}{2} + \mathcal{O}(x)}{1 + \mathcal{O}(x)} = -\frac{1}{2}$$

B 39

$$b) = \frac{a^2}{b^2}$$

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$$\boxed{\lim_{x \to 0} \frac{\sin(x) - 3\cos(x) + 2 + e^x}{x^2 + \tan(x)}}$$

Ordnung

a) $\sin(x) \underset{x_0 = 0}{=} 0 + \frac{1}{1!} x + \frac{0}{2!} x^2 + \frac{-1}{3!} x^3 + 0 + \sigma(x^5)$

$$\cos(x) = 1 + 0 + \frac{-1}{2!} x^2 + 0 + \frac{1}{4!} x^4 + \sigma(x^6)$$

$$e^x = 1 + \frac{1}{1!} x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \sigma(x^5)$$

$$\tan(x) = 0 + \frac{1}{1!} x + 0 + \frac{2}{3!} x^3 + \sigma(x^4)$$

$$(1 + \tan^2)' = 2 \tan (1 + \tan^2) = 2 \tan + 2 \tan^3$$

$$(\dots)' = 2(1 + \cancel{\tan^2}) + 6 \cancel{\tan^2 (1 + \tan^2)}$$

Nebenrechnung $(\tan')$

$$\lim_{x \to 0} \frac{x - \frac{x^3}{6} + o(x^5) \cancel{- 3} + 3\frac{x^2}{2} - 3 \frac{1}{24} x^4 \cancel{+ 2} \cancel{+ 1} + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}}{x^2 + \cancel{\tan} + \frac{2}{6} x^3 + \sigma(x^4)}$$

$$\lim_{x \to 0} \frac{2x + 2x^2 + \sigma(x^3)}{x^2 + x + \sigma(x^3)} = \lim_{x \to 0} \frac{2 + 2x + \sigma(x^2)}{1 + x + \sigma(x^2)}$$

$$\to 2$$