EMF I – Übung 05.02.10

H09)

Aufg. 1)

a)

$$\vec{k} = \omega\sqrt{\varepsilon_0\mu_0}\,\vec{e}_z \quad ; \quad |\vec{k}| = k$$

$$\underline{\vec{E}} = \underline{\vec{E}}_0 e^{-jkz} = E_{re} e^{j\varphi} e^{-jkz} \vec{e}_x$$

$$= E_{re}\left[\cos(\varphi) + j\sin(\varphi)\right] \vec{e}_x \cdot e^{-jkz}$$

RB: $\operatorname{Re}\{\underline{\vec{E}} e^{j\omega t}\}\big|_{t=0,z=0} \overset{!}{=} 0$

$$\Rightarrow \quad \cos(\varphi) = 0$$

$$\Rightarrow \quad \varphi = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$$

$$\varphi = \frac{\pi}{2}$$ , da kleinste pos. Lösung (Aufgabenstellung)

$$\underline{\vec{E}} = j E_{re} \vec{e}_x e^{-jkz}$$

$$\vec{E} = -E_{re} \vec{e}_x \cdot \sin(\omega t - kz)$$

b)

$$\underline{\vec{H}} = \frac{1}{Z}(\vec{e}_z \times \underline{\vec{E}}) \qquad \text{wg TEM-Welle}$$

$$\underline{\vec{H}} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, j E_{re} e^{-jkz} \vec{e}_y$$

$$\vec{H} = -\sqrt{\frac{\varepsilon_0}{\mu_0}}\, E_{re} \vec{e}_y \sin(\omega t - kz)$$

c)

$$\Phi_L(t) = \int_{F_L} \mu_0 \vec{H}\, d\vec{F} \qquad d\vec{F} = dx\, dz\, \vec{e}_y$$

$$= -\int_0^a dx\, \mu_0 \int_a^{2a} \sqrt{\frac{\varepsilon_0}{\mu_0}} E_{re} \sin(\omega t - kz)\, dz$$

$$= -a\sqrt{\frac{\varepsilon_0}{\mu_0}} E_{re} \mu_0 \int_a^{2a} \sin(\omega t - kz)\, dz$$

$$= -a\sqrt{\frac{\varepsilon_0}{\mu_0}} E_{re} \mu_0 \left[\frac{1}{k}\cos(\omega t - kz)\right]_a^{2a}$$

$$= -\frac{a E_{re}}{\omega}\left(\cos(\omega t - 2ka) - \cos(\omega t - ka)\right)$$

d)

$$\oint_L \vec{E}_{ind}\, d\vec{s} = -\dot{\Phi}_L(t)$$

$$= -\frac{a}{\omega} E_{re} \cdot \omega \left[ \sin(\omega t - 2ka) - \sin(\omega t - ka) \right]$$

$$= -a\, E_{re} \left( \sin(\omega t - 2ka) - \sin(\omega t - ka) \right)$$

e)

$$\vec{E} = -E_{re}\, \vec{e}_x \sin(\omega t - kz)$$

$$\oint_L \vec{E}\, d\vec{s} = a \cdot (-E_{re}) \left( \sin(\omega t - 2ka) - \sin(\omega t - ka) \right)$$

f)

$\oint_L \vec{E}_{ind}\, d\vec{s}$ verschwindet, wenn beide sin-Argumente $2\pi \cdot n$ Phasendifferenz haben wobei $n \in \mathbb{N}^+$

$$\Rightarrow \quad k \cdot a = n \cdot 2\pi$$

$$\Rightarrow \quad \omega \sqrt{\varepsilon_0 \mu_0} = \frac{n\, 2\pi}{a}$$

$$\Leftrightarrow \quad \omega = \frac{n \cdot 2\pi}{a \sqrt{\varepsilon_0 \mu_0}} \qquad n \in \mathbb{N}^+$$

$$\lambda = \frac{c}{f} = \frac{2\pi}{\omega} \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = \frac{a}{n}$$

„n ganze Wellenlängen passen in a rein"

Aufg. 2)

HR 1: $\sigma_1 = 0$, $\varepsilon_0, \mu_0$

HR 2: $\sigma_2 \to \infty$

$\vec{n}_e = \frac{1}{\sqrt{2}}(1,-1,0) \qquad \alpha = \frac{\pi}{4}$

a)

- Linear polarisiert, weil Phase zwischen x und y komponente gleich Null
- weil TEM:

$$|\vec{k}_e| = k = \omega\sqrt{\varepsilon_0\mu_0} = \sqrt{a^2+(-a)^2} = a\sqrt{2}$$

$$\Rightarrow \quad a = \frac{k}{\sqrt{2}} = \omega\sqrt{\frac{\varepsilon_0\mu_0}{2}}$$

b)

$\vec{n} \sim \vec{E} \times \vec{H}$

$$\vec{E}_e = Z_F \; \vec{H}_e \times \vec{n} \qquad \text{(nur für TEM!)}$$

$$= \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{H_{0e}}{\sqrt{2}} e^{-j\vec{k}_e\vec{r}} (\vec{e}_x + \vec{e}_y) \times \frac{1}{\sqrt{2}}(\vec{e}_x - \vec{e}_y)$$

$$= -\sqrt{\frac{\mu_0}{\varepsilon_0}} H_{0e} e^{-j\vec{k}_e\vec{r}} \vec{e}_z$$

Mögl. 1: $\boxed{E_{tan}|_{x=0} = 0}$

Mögl. 2: $\vec{H} = -\frac{1}{j\omega\mu} \operatorname{rot} \vec{E} = \frac{j}{\omega\mu} \operatorname{rot} \vec{E}$

$$\begin{pmatrix} H_{tan1} \\ H_{tan2} \\ H_n \end{pmatrix} = \frac{j}{\omega\mu} \begin{pmatrix} \partial_{tan1} \\ \partial_{tan2} \\ \partial_n \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ E_n \end{pmatrix} \quad \Rightarrow \quad \boxed{H_n|_{x=0} = 0}$$

Außerdem:

$$\begin{pmatrix} E_{tan1} \\ E_{tan2} \\ E_n \end{pmatrix} = \frac{1}{j\omega\varepsilon} \begin{pmatrix} \partial_{tan1} \\ \partial_{tan2} \\ \partial_n \end{pmatrix} \times \begin{pmatrix} H_{tan1} \\ H_{tan2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ E_n \end{pmatrix} \quad \Rightarrow \quad \boxed{\frac{\partial}{\partial n} \vec{H}_t = 0}$$

$H_n|_{x=0} = 0$

$$\Rightarrow \vec{H}_e(\vec{r}) = \frac{H_{0e}}{\sqrt{2}} e^{-j\omega\sqrt{\frac{\varepsilon_0\mu_0}{2}}(-y)} (\vec{e}_x + \vec{e}_y)$$

$$H_n = H_{e_x}(x=0) + H_{r_x}(x=0)$$

$$= \frac{H_{0e}}{\sqrt{2}} e^{j\frac{k}{\sqrt{2}}y} - H_{0r} e^{j\frac{k}{\sqrt{2}}y} \stackrel{!}{=} 0$$

$$\Rightarrow H_{0r} = \frac{H_{0e}}{\sqrt{2}}$$

$$\vec{E}_r(\vec{r}) = \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{H_{0e}}{\sqrt{2}} e^{-j\vec{k}_r\vec{r}} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}}$$

$$= \sqrt{\frac{\mu_0}{\varepsilon_0}} H_{0e} \vec{e}_z e^{-j\vec{k}_r\vec{r}}$$

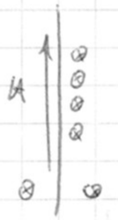
Überprüfung: $E_{tan}(x=0) = 0$

$$= E_{r_z}(x=0) + E_{e_z}(x=0)$$

$$= \sqrt{\frac{\mu_0}{\varepsilon_0}} H_{0e} \vec{e}_z \left(-e^{-j\left(\frac{-k}{\sqrt{2}}\right)y} + e^{-j\left(-\frac{k}{\sqrt{2}}\right)y}\right)$$

$$= 0 \quad \checkmark$$

c)



$$\vec{H} = -\vec{n} \times \vec{J}_F$$

$$\Leftrightarrow \vec{J}_F = \vec{n} \times \vec{H}_{tan}|_{x=0}$$

$$H_{tan} = \frac{H_{0e}}{\sqrt{2}} e^{+j\frac{k}{\sqrt{2}}y} (2\vec{e}_y)$$

$$J_F = \frac{H_{0e}}{\sqrt{2}} e^{j\frac{k}{\sqrt{2}}y} 2(-\vec{e}_x \times \vec{e}_y)$$

$$= -2\frac{H_{0e}}{\sqrt{2}} e^{j\frac{k}{\sqrt{2}}y} \vec{e}_z$$

$$= -\sqrt{2} H_{0e} e^{j\frac{k}{\sqrt{2}}y} \vec{e}_z$$

d) $\overline{\vec{P}(\vec{r})} = \frac{1}{2} Re\{ \underline{\vec{P}}(\vec{r}) \}$

$$\underline{\vec{P}}(\vec{r}) = \vec{E} \times \vec{H}^*$$

$$= \left( \sqrt{\frac{\mu_0}{\varepsilon_0}} H_{0e} \vec{e}_z \left( -e^{-j\frac{k}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}\begin{pmatrix}x\\y\\0\end{pmatrix}} + e^{-j\frac{k}{\sqrt{2}}\begin{pmatrix}-1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}x\\y\\0\end{pmatrix}} \right) \right) \times$$

$$\left( \frac{H_{0e}}{\sqrt{2}} \left( e^{-j\frac{k}{\sqrt{2}}(x-y)} (\vec{e}_x + \vec{e}_y) + e^{j\frac{k}{\sqrt{2}}(x+y)} (-\vec{e}_x + \vec{e}_y) \right)^* \right)$$

$$= \sqrt{\frac{\mu_0}{2\varepsilon_0}} |H_{0e}|^2 e^{j\frac{k}{\sqrt{2}}y} \left( e^{j\frac{k}{\sqrt{2}}x} - e^{-j\frac{k}{\sqrt{2}}x} \right) \left( e^{-j\frac{k}{\sqrt{2}}y} \right)$$

$$\cdot \left( e^{j\frac{k}{\sqrt{2}}x} (\vec{e}_y - \vec{e}_x) + e^{-j\frac{k}{\sqrt{2}}x} (-\vec{e}_y - \vec{e}_x) \right)$$

$$= \sqrt{\frac{\mu_0}{2\varepsilon_0}} |H_{0e}|^2 \, 2j \sin\left(\frac{kx}{\sqrt{2}}\right) \left( \vec{e}_y 2j \sin\left(\frac{kx}{\sqrt{2}}\right) - 2\vec{e}_x \cos\left(\frac{kx}{\sqrt{2}}\right) \right)$$

$$= -\sqrt{\frac{\mu_0}{2\varepsilon_0}} |H_{0e}|^2 4 \left[ \vec{e}_y \sin^2\left(\frac{kx}{\sqrt{2}}\right) + j \sin\left(\frac{kx}{\sqrt{2}}\right) \cos\left(\frac{kx}{\sqrt{2}}\right) \vec{e}_x \right]$$

e)

- Transport in neg. y-Richtung
- $TE_y$ - Welle

Aufg. 3) (Oberflächenwellen)

a) $TE_x$-Welle: $\vec{G}_i = \vec{0}$ mit $i = 1, 2$

($E_x = 0$) $\vec{F}_i = f_i(x,z)\,\vec{e}_x = \tilde{f}_i(x)\, e^{-j\beta z}\,\vec{e}_x$

mit $\beta^2 = \omega^2 \varepsilon_{eff} \mu_0$

wobei $\varepsilon_2 \le \varepsilon_{eff} \le \varepsilon_1$

$$\Delta f_i + k_i^2 f_i = 0 \quad \text{Helmholtz-Gl.}$$

$$\Rightarrow \tilde{f}_i'' + p_i^2 \tilde{f}_i = 0 \quad \text{mit } p_i^2 = k_i^2 - \beta^2 \text{ oder}$$

$$p_i^2 = (\varepsilon_{ri} - \varepsilon_{reff})\, k_0^2$$

wobei $k_0^2 = \omega^2 \varepsilon_0 \mu_0$

Mit $p_1^2 \ge 0$ und $\alpha_2^2 = -p_2^2 \ge 0$ folgt

$$\tilde{f}_1'' + p_1^2 \tilde{f}_1 = 0$$
$$\tilde{f}_2'' - \alpha_2^2 \tilde{f}_2 = 0$$

Allgemeine Lösungen: $\tilde{f}_1(x) = A_1 \cos(p_1 x) + B_1 \sin(p_1 x)$

$\tilde{f}_2(x) = A_2 e^{\alpha_2 x} + B_2 e^{-\alpha_2 x}$

b)

$$\vec{E}_i = \left(0, -\frac{\partial f_i}{\partial z}, 0\right)$$

$$\vec{H}_i = \frac{1}{j\omega\mu_0}\left(\left(k_i^2 + \frac{\partial^2}{\partial x^2}\right) f_i,\ 0,\ \frac{\partial^2 f_i}{\partial x \partial z}\right)$$

$$k_i^2 f_i + f_i'' = (k_i^2 - p_i^2) f_i = \beta^2 f_i$$

$$E_{y1} = -\frac{\partial f_1}{\partial z} = j\beta f_i = j\beta \cdot \tilde{f}_1 e^{-j\beta z} = j\beta\left(A_1 \cos(p_1 x) + B_1 \sin(p_1 x)\right) e^{-j\beta z}$$

$$E_{y2} = -\frac{\partial f_2}{\partial z} = j\beta \tilde{f}_2 e^{-j\beta z} = j\beta \cdot \left(A_2 e^{+\alpha_2 x} + B_2 e^{-\alpha_2 x}\right) e^{-j\beta z}$$

$$H_{x1} = \frac{\beta^2}{j\omega\mu_0} f_1 = \frac{\beta^2}{j\omega\mu_0} \tilde{f}_1 e^{-j\beta z} \qquad H_{x2} =$$

$$H_{z1} = \frac{1}{j\omega\mu_0} \frac{\partial^2 f_1}{\partial x \partial z} = -\frac{\beta}{\omega\mu_0} \frac{\partial f_1}{\partial x} = \frac{\beta p_1}{\omega\mu_0}\left(A_1 \sin(p_1 x) - B_1 \cos(p_1 x)\right) e^{-j\beta z}$$

$$H_{z2} = \dots = -\frac{\beta\alpha_2}{\omega\mu_0}\left(A_2 e^{\alpha_2 x} - B_2 e^{-\alpha_2 x}\right) e^{-j\beta z}$$

c)

elektr. Wand: bei $x=0$: $\vec{E}_{tan} = \vec{0}$, $\underline{H}_n = 0$

bzw. $\frac{\partial}{\partial n} \vec{\underline{H}}_{tan} = \vec{0}$

$$\underline{E}_{y1}(x=0) \overset{!}{=} 0 \quad \Rightarrow \quad \underline{A}_1 = 0$$

$$\underline{E}_{y2}(x \to \infty) \overset{!}{=} 0 \quad \Rightarrow \quad \underline{A}_2 = 0$$

d)

Grenzfläche zweier Dielektrika bei $x=h$:

$\vec{\underline{E}}_{tan}$ stetig, $\vec{\underline{H}}_{tan}$ stetig (($H_n$ stetig, da $\mu$ keinen Sprung macht)

1.) $\underline{E}_{y1}(x=h) \overset{!}{=} \underline{E}_{y2}(x=h)$

2.) $\underline{H}_{z1}(x=h) \overset{!}{=} \underline{H}_{z2}(x=h)$

$$\Rightarrow \quad \underline{B}_1 \sin(p_1 h) = \underline{B}_2 e^{-\alpha_2 h}$$
$$-p_1 \underline{B}_1 \cos(p_1 h) = \alpha_2 \underline{B}_2 e^{-\alpha_2 h}$$

$$\Rightarrow \quad \tan(p_1 h) = -\frac{p_1}{\alpha_2} \quad \text{(char. Gleichung)}$$

e)

$$\tan(p_1 h) = -\sqrt{3}$$

Hinweis: $n=1$ (niedrigste Mode) $\Rightarrow p_1 h = \pi - \frac{\pi}{3} = \frac{2}{3}\pi$

↑ kleinster pos. Eigenwert

$$\Rightarrow \quad p_1 = \frac{2\pi}{3h} \quad \text{und} \quad \alpha_2 = \frac{p_1}{\sqrt{3}} = \frac{2\pi}{3\sqrt{3}\, h}$$

$$p_1^2 = k_1^2 - \beta^2 = \omega^2 \mu_0 (\varepsilon_1 - \varepsilon_{eff}) \geq 0$$
$$\alpha_2^2 = \beta^2 - k_2^2 = \omega^2 \mu_0 (\varepsilon_{eff} - \varepsilon_2) \geq 0$$

$$\Rightarrow \quad p_1^2 + \alpha_2^2 = \omega^2 \mu_0 (\varepsilon_1 - \varepsilon_2) \qquad \alpha_2^2 - p_1^2 = \omega^2 \mu_0 (2\varepsilon_{eff} - \varepsilon_1 - \varepsilon_2)$$

$$\Rightarrow \quad \omega^2 = \frac{p_1^2 + \alpha_2^2}{\mu_0(\varepsilon_1 - \varepsilon_2)} \qquad \varepsilon_{eff} = \underbrace{\frac{\alpha_2^2 - p_1^2}{\alpha_2^2 + p_1^2}}_{-1 \leq \nu \leq 1} \frac{\varepsilon_1 - \varepsilon_2}{2} + \frac{\varepsilon_1 + \varepsilon_2}{2}$$

$$\omega^2 = \frac{16\pi^2}{27 h^2} \frac{1}{\mu_0(\varepsilon_1 - \varepsilon_2)} \qquad \varepsilon_{eff} = \varepsilon_2 + \frac{1}{4}(\varepsilon_1 - \varepsilon_2)$$

f)

Grenzfrequenz für $\alpha = 0$ bzw $\varepsilon_{eff} = \varepsilon_2$

(Felder klingen in x-Richtung nicht mehr ab)

$$\tan(p_1 h) \to -\infty \quad \Rightarrow \quad p_1 h \underset{\uparrow}{=} \frac{\pi}{2}$$

kleinster Eigenwert $p_1 > 0$
(niedrigster Mode)

$$\Rightarrow \quad p_1 = \frac{\pi}{2h} \quad , \quad \varepsilon_{eff} = \varepsilon_2$$

$$f_C = \frac{1}{4h} \frac{1}{\sqrt{\mu_0(\varepsilon_1 - \varepsilon_2)}}$$