Frühjahr 2005

Aufgabe 1:

(a) $\vec{B}(\vec{r})$; $\vec{r} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ für $x < a(t)$
$y < a(t)$

Induktion eines unendlich langen Leiters: (Ebene $z=0$)

$$\vec{B}_{1,2}(\vec{r}) = \frac{\mu_0 \cdot I}{2\pi\, d(\vec{r})} \cdot \vec{e}_z$$ mit $d(\vec{r}) \hat{=}$ Abstand zum Leiter

$$\vec{B}_1(\vec{r}) = \frac{\mu_0 \cdot I}{2\pi} \cdot \vec{e}_z \cdot \frac{1}{a(t)-x}$$

$$\vec{B}_2(\vec{r}) = \frac{\mu_0 \cdot I}{2\pi} \cdot \vec{e}_z \cdot \frac{1}{a(t)-y}$$

somit

$$\vec{B}(\vec{r}) = \vec{B}_1(\vec{r}) + \vec{B}_2(\vec{r}) = \frac{\mu_0 \cdot I}{2\pi} \cdot \vec{e}_z \cdot \left(\frac{1}{a(t)-x} + \frac{1}{a(t)-y} \right)$$

(b) Bestimme $\phi(t)$; Fluss eines unendlichen Leiters $\times 2$
(wegen Symmetrie)

$$\phi(t) = \int_{-b}^{b}\int_{-b}^{b} \vec{B}(\vec{r}) \cdot \vec{e}_z \, dx \cdot dy$$

$$= \frac{\mu_0 \cdot I}{2\pi} \cdot 2 \cdot \int_{-b}^{b}\int_{-b}^{b} \frac{1}{a(t)-x} \, dx \, dy$$

$$= \frac{\mu_0 \cdot I}{\pi} \cdot 2b \cdot \int_{-b}^{b} \frac{1}{a(t)-x} \, dx$$

$$= \frac{2\mu_0 b \cdot I}{\pi} \cdot \Big[-\ln\,(a(t)-x) \Big]_{-b}^{b}$$

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$$= \frac{2\mu_0 b \cdot I}{\pi} \cdot \left(-\ln(a(t)-b) + \ln(a(t)+b)\right)$$

$$= \frac{2\mu_0 b \cdot I}{\pi} \cdot \ln\left(\frac{a(t)+b}{a(t)-b}\right)$$

(c) $u(t) = -\frac{d\phi(t)}{dt}$ mit $\frac{da(t)}{dt} = v$

$$\Rightarrow \quad u(t) = -\frac{2\mu_0 \cdot b \cdot I}{\pi} \cdot \frac{d}{dt}\left(\ln(a(t)+b) - \ln(a(t)-b)\right)$$

$$= -2\mu_0 b \cdot \frac{I}{\pi}\left(v \cdot \frac{1}{a(t)+b} - v \cdot \frac{1}{a(t)-b}\right)$$

$$= 2\mu_0 I v \frac{b}{\pi} \cdot \left(\frac{1}{a(t)-b} - \frac{1}{a(t)+b}\right)$$

(d) $u(t) \equiv 0$ wenn $\frac{d\phi}{dt} = 0 \Rightarrow \phi = \text{const}$

aus (b) $\phi = 2\mu_0 \frac{b}{\pi} \cdot I(t) \cdot \ln\left(\frac{b + a(t)}{-b + a(t)}\right) = \text{const} \mathrel{\hat{=}} K$

$$\Rightarrow \quad I(t) = \frac{K \cdot \pi}{2\mu_0 b} \cdot \frac{1}{\ln\left(\frac{b+a(t)}{-b+a(t)}\right)}$$

$$I(t) = I_0 \cdot \ln\left(\frac{a(t)+b}{a(t)-b}\right)^{-1}$$

mit I_0 beliebiger Strom

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Aufgabe 2:

a) $\underline{\vec{E}}(\rho, z) = E(\rho) \cdot \exp(-j k_0 z) \cdot \vec{e}_\rho = E_\rho \cdot \vec{e}_\rho$

Helmholtzgleichung: $\boxed{(\Delta + k_0^2) \cdot \underline{\vec{E}}(\rho, z) = \vec{0}}$ (*)

Da $\underline{\vec{E}}$ nur Fkt. von ρ und z und $\underline{\vec{E}} \parallel \vec{e}_\rho$ ist:

$$\Delta \underline{\vec{E}}(\rho, z) = \left\{ \frac{\partial}{\partial \rho}\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \cdot E_\rho) \right) + \frac{\partial^2 E_\rho}{\partial z^2} \right\} \cdot \vec{e}_\rho + 0$$

Δ in Zylinderkoordinaten nach Formelsammlung

$$= \left\{ \frac{\partial}{\partial \rho}\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \cdot E_\rho) \right) - k_0^2 \cdot E_\rho \right\} \cdot \vec{e}_\rho$$

(*): $\vec{0} = (\Delta + k_0^2)\, \underline{\vec{E}}(\rho, z) \Rightarrow 0 = \frac{\partial}{\partial \rho} \underbrace{\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \cdot E_\rho) \right)}_{=0}$ (**)

$\boxed{\vec{\nabla}\, \underline{\vec{E}}(\rho, z) = 0}$ $\Rightarrow \longrightarrow = 0$

Formelsammlung (***)

(sonst wäre $E_\rho \sim \rho$ auch Lsg. für (**))

(**) & (***) $\Rightarrow 0 = \frac{\partial}{\partial \rho}(\rho \cdot E_\rho) \Rightarrow E_\rho \sim \frac{1}{\rho}$ also $\boxed{E(\rho) \sim \frac{1}{\rho}}$

P_1, P_2 Punkte auf der Oberfläche vom Innenleiter, bzw. Innenseite des Außenleiters

$$U_0 = \int_{P_1}^{P_2} \underline{\vec{E}}(\rho, z)\, d\vec{r} = \int_a^b E(\rho)\, d\rho = \int_a^b \frac{A}{\rho}\, d\rho = A \cdot \ln \frac{b}{a}$$

$$\Rightarrow A = \frac{U_0}{\ln \frac{b}{a}} \quad \Rightarrow E(\rho) = \frac{U_0}{\ln\left(\frac{b}{a}\right)} \cdot \frac{1}{\rho}$$

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(b) $\vec{\nabla} \times \vec{E} = -j\omega\mu_0 \vec{H} \qquad \left(= -\mu_0 \frac{\partial}{\partial t} \vec{H}\right)$

$$\Rightarrow \vec{\nabla} \times \underline{\vec{E}}(\varrho, z) = -j\omega\mu_0 \underline{\vec{H}}(\varrho, z)$$

$$\vec{\nabla} \times \underline{\vec{E}}(\varrho, z) = \emptyset + \vec{e}_\varphi \cdot \left\{ \frac{\partial}{\partial t} \left(E(\varrho) \cdot \exp(-jk_0 z) \right) - \emptyset \right\} + \emptyset$$

$$= \vec{e}_\varphi \cdot \left\{ -jk_0 E(\varrho) \exp(-jk_0 z) \right\} = -j\omega\mu_0 \underline{\vec{H}}(\varrho, z)$$

$$\underline{\vec{H}}(\varrho, z) = \frac{k_0}{\omega\mu_0} E(\varrho) \exp(-jk_0 z) \cdot \vec{e}_\varrho = \sqrt{\frac{\varepsilon_0}{\mu_0}} \cdot \frac{U_0}{\ln\left(\frac{b}{a}\right)} \cdot \frac{1}{\varrho} \cdot \exp(-jk_0 z) \cdot \vec{e}_\varphi$$

(c) $$\underline{U}_+(z) = \int_{P_1'}^{P_2'} \underline{\vec{E}}(\varrho, z)\, d\vec{r} = \int_{P_1'}^{P_2'} \frac{U_0}{\ln\left(\frac{b}{a}\right)} \exp(-jk_0 z) \cdot \frac{1}{\varrho}\, d\varrho$$

$$= \frac{U_0}{\ln\left(\frac{b}{a}\right)} \exp(-jk_0 z) \underbrace{\int_a^b \frac{1}{\varrho}\, d\varrho}_{= \ln \frac{b}{a}}$$

$$\underline{U}_+(z) = U_0 \cdot \exp(-jk_0 z)$$

$$\underline{I}_+(z) = \oint_{C_0} \underline{\vec{H}}\, d\vec{r} = \int_0^{2\pi} \underline{\vec{H}}(\varrho = a, z) \cdot a \cdot d\varphi$$

$\hookrightarrow C_0$ z.B. Kurve entlang der Oberfläche des Innenleiters
($\Rightarrow \varrho = a$, $\underline{\vec{H}}\, d\vec{r} = \emptyset + (\vec{H} \cdot \vec{e}_\varphi)\, a \cdot d\varphi$)

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{2\pi}{\ln \frac{b}{a}} U_0 \cdot \exp(-jk_0 z)$$

$$Z_L = \frac{\underline{U}_+(z)}{\underline{I}_+(z)} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{\ln\left(\frac{b}{a}\right)}{2\pi}$$

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(d) $z = \emptyset$: $\vec{E}, \vec{H}$ stetig

$z = h$: $\vec{E} \equiv \emptyset$, $\vec{e}_z \times (\emptyset - \underline{\vec{H}}_2) = \vec{J}_F$

$z = \emptyset$: $\underline{u}(z), \underline{i}(z)$ stetig

$z = h$: $\underline{u}(h) \equiv \emptyset$

(e) $z \leq \emptyset$: $\underline{U}(z) = U_0 \exp(-jk_0 z) + U_{0-} \exp(jk_0 z)$

$$\underline{i}(z) = \frac{U_0}{Z_L} \exp(-jk_0 z) - \frac{U_{0-}}{Z_L} \exp(jk_0 z)$$

$\emptyset < z < h$: $\underline{U}_2(z) = U_{2+} \exp(-jkz) + U_{2-} \exp(jkz)$

$$\underline{i}_2(z) = \frac{U_{2+}}{Z_{L2}} \exp(-jkz) - \frac{U_{2-}}{Z_{L2}} \exp(jkz)$$

Grenzbed. von (d) : $\underline{U}_2(h) \equiv \emptyset$ (*)

$\underline{U}(\emptyset) = \underline{U}_2(\emptyset)$ (**)

$\underline{i}(\emptyset) = \underline{i}_2(\emptyset)$ (***)

(*) $\Rightarrow U_{2-} = -U_{2+} \exp(-2jkh)$

$$\Rightarrow \underline{U}_2(z) = U_{2+} \exp(-jkh) \cdot \left(\exp(-jk[z-h]) - \exp(jk[z-\right.$$

(**) $\cancel{U_{2+}}$ $U_0 + U_{0-} = U_{2+} \underbrace{(1 - \exp(-2jkh))}_{=:\alpha}$

(***) $\Rightarrow \frac{U_0}{Z_L} - \frac{U_{0-}}{Z_L} = \frac{U_{2+}}{Z_{L2}} (1 + \exp(-2jkh)) \Rightarrow U_0 - U_{0-} =$

$$U_{2+} \underbrace{\frac{Z_L}{Z_{L2}} (1 + \exp(-2jkh))}_{=:\beta}$$

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$$(**)\ \&\ (***) \Rightarrow \frac{U_0 + U_{0-}}{\alpha} = \frac{U_0 - U_{0-}}{\beta} \Rightarrow U_{0-} = U_0 \frac{\alpha-\beta}{\alpha+\beta}$$

$$\sim \quad \Rightarrow 2U_0 = U_{2+}\,(\alpha+\beta) \Rightarrow U_{2+} = U_0 \cdot \frac{2}{\alpha+\beta}$$

$$U_{2-} = -U_0 \cdot \frac{2\exp(-2jkh)}{\alpha+\beta}$$

Reflexionskoeff. $\Gamma = \frac{U_{0-}}{U_0} = \frac{\alpha-\beta}{\alpha+\beta} = \frac{Z_{L2}\ j\sin(kh) - Z_L\cos(kh)}{Z_{L2}\ j\sin(kh) + Z_L\cos(kh)}$

$$\left(\alpha = \exp(-jkh)\cdot\left\{2j\sin(kh)\right\};\ \beta = \exp(-jkh)\left\{2\cos(kh)\right\}\cdot\frac{Z_L}{Z_{L2}}\right)$$

$$\underline{U}(z) = U_0\cdot\exp(-jk_0 z) + \Gamma\cdot U_0\cdot\exp(jk_0 z)$$

$$\underline{i}(z) = \frac{U_0}{Z_L}\exp(-jk_0 z) - \frac{\Gamma U_0}{Z_L}\exp(jk_0 z)$$

$$\underline{U}_2(z) = \frac{2U_0}{[1-\exp(-2jkh)] + \left[\frac{Z_L}{Z_{L_2}}(1+\exp(-2jkh))\right]} \cdot \left\{\exp(-jkz) - \exp(jkz - 2jkh)\right\}$$

$$\underline{i}_2(z) = \frac{2U_0}{Z_{L2}[1-\exp(-2jkh)] + Z_L[1+\exp(-2jkh)]} \cdot \left\{\exp(-jkz) + \exp(jkz - 2jkh)\right\}$$

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F05

Lösung A3

a) $\Delta \underline{F}(\rho,z) + k^2 \underline{F}(\rho,z) = 0 \; ; \quad \Delta \underline{G}(\rho,z) + k^2 \underline{G}(\rho,z) = 0 \; ; \quad k^2 = \omega^2 \varepsilon \mu$ (1)

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho} \underline{f}(\rho) e^{-j\beta z}\right) + \frac{\partial^2}{\partial z^2}\left(\underline{f}(\rho) e^{-j\beta z}\right) + k^2 \underline{f}(\rho) e^{-j\beta z} = 0 \quad (2)$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \underline{f}(\rho)}{\partial \rho}\right) + p^2 \underline{f}(\rho) = 0 \quad ; \quad p^2 = k^2 - \beta^2 \quad (3)$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \underline{g}(\rho)}{\partial \rho}\right) + p^2 \underline{g}(\rho) = 0 \quad ; \quad \text{analog} \quad (4)$$

(3), (4): Besselsche DGL nullter Ordnung ($p \neq 0$ nach Vorau

Lösungen:

$$\underline{f}(\rho) = A\, J_0(p\rho) + B\, N_0(p\rho) \quad (5)$$

$$\underline{g}(\rho) = C\, J_0(p\rho) + D\, N_0(p\rho) \quad (6)$$

b) Die Felder und damit $\underline{f}, \underline{g}$ dürfen für $\rho \to 0$ nicht singulär werden

$$\Rightarrow \boxed{B = 0 \, , \quad D = 0} \quad (7)$$

Grenzbedingung auf G_1 (elektr. Wand; $\varphi = 0$, $\varphi = \pi$):

$\underline{E}_{tang} = 0 \quad \Rightarrow \quad \alpha) \; \underline{E}_z = 0 \qquad \beta) \; \underline{E}_\rho = 0$

$\Downarrow$

$\underline{g}(\rho) = 0 \quad (C = 0)$

erfüllt!

$\Downarrow$

$$\frac{d\underline{g}(\rho)}{d\rho} = 0 \quad \Rightarrow \quad \underline{E}_\rho = 0 \quad (8)$$

überall, also auch auf G_1

Somit: $\underline{f}(\rho) = A\, J_0(p,\rho) \; ; \quad \underline{g}(\rho) = 0$ (9)

c) Grenzbedingung auf G_2 (magn. Wand; $\rho = a$):

$\underline{H}_{tang} = 0 \quad \Rightarrow \quad \alpha) \quad \underline{H}_z = 0 \qquad\qquad \beta) \quad \underline{H}_\varphi = 0$

$\Rightarrow \dfrac{d\underline{\Phi}}{d\rho} = 0$ erfüllt wegen (8)

$\Downarrow$

(10) $\quad \underline{f}(\rho = a) = 0$

$\Downarrow$

(11) $\quad J_0(pa) = 0$

Die Gleichung (11) legt die möglichen Werte von p fest; da die Nullstellen der Besselfunktion reell, abzählbar sind, gilt das auch für die p

$$p = p_1, p_2, p_3, \dots, p_\ell, \dots \qquad p_\ell \text{ reell} > 0 \qquad (12)$$

Aus $k^2 - \beta^2 = p^2$ folgt $k^2 - \beta_\ell^2 = p_\ell^2$

also $\quad \beta_\ell = \sqrt{k^2 - p_\ell^2}$

$$= \sqrt{\omega^2 \varepsilon \mu_0 - p_\ell^2} \quad ; \quad \ell = 1, 2, 3, \dots \qquad (13)$$

d) Es gibt nur Wellenausbreitung in $+z$-Richtung für $\mathrm{Re}\{\beta_\ell\} > 0$ (14)

aus (13): $\quad \mathrm{Re}\{\beta_\ell\} > 0$, falls $\omega^2 \varepsilon \mu_0 > p_\ell^2$

~~Grenzbedingung~~

$\Rightarrow$ Grenzfrequenz für Wellentyp "ℓ": $\quad \omega_\ell = \dfrac{p_\ell}{\sqrt{\varepsilon \mu_0}}$ (15)

$$v_{Gr,\ell} = \frac{\partial \omega}{\partial \beta_\ell} \; ; \quad (13) \Rightarrow \frac{1}{\sqrt{\varepsilon \mu_0}} \sqrt{\beta_\ell^2 + p_\ell^2} = \omega \qquad (16)$$

$$v_{Gr,\ell} \overset{(16)}{=} \frac{1}{\sqrt{\varepsilon \mu_0}} \frac{\beta_\ell}{\sqrt{\beta_\ell^2 + p_\ell^2}} \overset{(16)}{=} \frac{\beta_\ell}{\varepsilon \mu_0 \omega} = \frac{1}{\varepsilon \mu_0} \sqrt{\varepsilon \mu_0 - \left(\frac{p_\ell}{\omega}\right)^2}$$

$$v_{Gr,e} = \frac{1}{\sqrt{\varepsilon \mu_0}} \sqrt{1 - \left(\frac{\omega_e}{\omega}\right)^2} \quad (17)$$

e) $$\underline{\vec{E}}_e = + \frac{df}{d\rho} e^{-\beta_e z} \vec{e}_\varphi$$

$$= \vec{e}_\varphi A_e p_e J_0'(p_e \rho) e^{-j\beta_e z} \; ; \quad A_e = A_e^* \text{ o.B.d.A.} \quad (18)$$

$$\underline{\vec{H}}_e = \vec{e}_z \frac{p_e^2 A_e}{j\omega\mu_0} J_0(p_e \rho) e^{-j\beta_e z} - \frac{\beta_e p_e A_e}{\omega\mu_0} J_0'(p_e \rho) e^{-j\beta_e z} \vec{e}_\rho \quad (19)$$

$$\overline{\vec{S}_e(t)} = \frac{1}{2} \mathrm{Re} \left\{ \underline{\vec{E}}_e \times \underline{\vec{H}}_e^* \right\}$$

$$= \frac{1}{2} \mathrm{Re} \Big\{ \underbrace{\frac{A_e^2 p_e^3}{\omega\mu_0} j J_0(p_e\rho) J_0'(p_e\rho) \vec{e}_\varphi \times \vec{e}_z}_{\text{rein Imaginär}}$$

$$- \frac{A_e^2 p_e^2 \beta_e}{\omega\mu_0} \left[J_0'(p_e\rho) \right]^2 \underbrace{\vec{e}_\varphi \times \vec{e}_\rho}_{-\vec{e}_z} \Big\}$$

$$\overline{\vec{S}(t)} = \frac{A_e^2 p_e^2 \beta_e}{2\omega\mu_0} \left[J_0'(p_e\rho) \right]^2 \vec{e}_z \quad (20)$$

$$z = 0 \Rightarrow e^{-j\beta z} = 1$$

$$t = \frac{T}{8} \Rightarrow \omega t = \frac{2\pi}{T} \frac{T}{8} = \frac{\pi}{4} \Rightarrow e^{j\omega t} = \frac{1}{\sqrt{2}}(1+j) \quad (21)$$

$$\vec{E}_e(t) \Big|_{\substack{z=0 \\ t=\frac{T}{8}}} =: \vec{E}_e\left(0, \frac{T}{8}\right) = \mathrm{Re}\left\{ \underline{\vec{E}}_e e^{j\omega t} \right\}_{\substack{z=0 \\ t=\frac{T}{8}}}$$

$$= \vec{e}_\varphi A_e p_e J_0'(p_e\rho) \frac{1}{\sqrt{2}} \quad (22)$$

$$\vec{H}_e(t)\Big|_{\substack{z=0\\ t=\frac{T}{8}}} =: \vec{H}_e\left(0, \frac{T}{8}\right) = \mathrm{Re}\left\{\underline{H}_e\, e^{j\omega t}\right\}_{\substack{z=0\\ t=\frac{T}{8}}}$$

$$= \frac{p_e^2 A_e}{\omega \mu_0 \sqrt{2}} J_0(p_e \rho)\, \vec{e}_z$$

$$- \frac{\beta_e\, p_e A_e}{\omega \mu_0 \sqrt{2}} J_0'(p_e \rho)\, \vec{e}$$

$$\vec{S}_e\Big|_{\substack{z=0\\ t=\frac{T}{8}}} = \vec{E}_e\left(0, \frac{T}{8}\right) \times \vec{H}_e\left(0, \frac{T}{8}\right)$$

$$= \frac{p_e^3 A_e^2}{2\omega\mu_0} J_0'(p_e\rho)\, J_0(p_e\rho)$$

$$+ \frac{A_e^2 p_e^2 \beta_e}{2\omega\mu_0}\left[J_0'(p_e\rho)\right]^2 \vec{e}_z \quad (23.$$