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a.)

λ : Wellenlänge

$4\pi d^2$: Kugeloberfläche

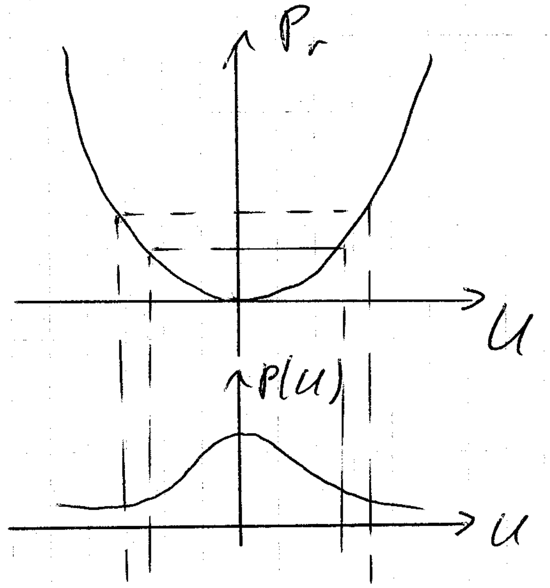
$$P_r = T(U) = P_t \cdot \frac{G_t G_r \lambda^2}{(4\pi d)^2}$$

$$= \frac{U^2}{R} \cdot \frac{G_t G_r \lambda^2}{(4\pi d)^2}$$

$$= \beta^2 U^2 \qquad \beta \geq 0$$

Die Abbildung ist für $u \in \mathbb{R}$ nicht injektiv.

b.)



$$\mathbb{R} = \mathbb{R}_- \cup \mathbb{R}_+^0 = \underbrace{(-\infty, 0)}_{S_1} \cup \underbrace{[0, \infty)}_{S_2}$$

T_1 und T_2 injektiv und stetig diffbar

$$\underset{u<0}{T_1(U)} = U^2\beta^2 \qquad \underset{u\geq 0}{T_2(U)} = U^2\beta^2$$

$$T_1^{-1}(p_r) = -\frac{1}{\beta}\sqrt{p_r} = U$$

$$T_2^{-1}(\rho_r) = \frac{1}{\beta}\sqrt{\rho_r} = u$$

$$\left|\frac{\partial T_1^{-1}(\rho_r)}{\partial \rho_r}\right| = \frac{1}{2\beta\sqrt{\rho_r}} = \left|\frac{\partial T_2^{-1}(\rho_r)}{\partial \rho_r}\right|$$

$$f_u(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{u^2}{2\sigma^2}}$$

$$f_{\rho_r}(\rho_r) = \sum_{k=1}^{2} f_u(T_k^{-1}(\rho_r))\left|\frac{\partial T_k^{-1}(\rho_r)}{\partial \rho_r}\right|$$

$$= \Big[\frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{\rho_r}{2\beta^2\sigma^2}} \cdot \frac{1}{2\beta\sqrt{\rho_r}} \mathbb{I}_{[0,\infty)}(\rho_r)$$

$$+ \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\rho_r}{2\beta^2\sigma^2}} \frac{1}{2\beta\sqrt{\rho_r}} \mathbb{I}_{[0,\infty)}(\rho_r)\Big]$$

$$= \frac{1}{\sqrt{2\pi\rho_r\beta^2\sigma^2}} e^{-\frac{\rho_r}{2\beta^2\sigma^2}} \mathbb{I}_{[0,\infty)}(\rho_r)$$

$f_{\rho_r}(\rho_r)$ ist eine Wahrscheinlichkeitsdichtefkt., da i.) $f_{\rho_r}(\rho_r) \geq 0$ für alle $\rho_r > 0$

ii.) $$\int_{-\infty}^{\infty} f_{\rho_r}(\rho_r)\, d\rho_r = \int_0^{\infty} \frac{1}{\sqrt{2\pi\rho_r\beta^2\sigma^2}} e^{-\frac{\rho_r}{2\beta^2\sigma^2}} d\rho_r$$

subst: $x = \sqrt{\rho_r}$

$\frac{dx}{d\rho_r} = \frac{1}{2\sqrt{\rho_r}}$

$$= \int_0^{\infty} \frac{2}{\sqrt{2\pi\beta^2\sigma^2}} \exp\left(-\frac{x^2}{2\beta^2\sigma^2}\right) dx$$ (symme Ktte)

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\beta^2\sigma^2}} \exp\left(-\frac{x^2}{2\beta^2\sigma^2}\right) dx$$ (Dichte von $x \sim \mathcal{N}(0, \beta^2 \sigma^2)$)

$$\underline{\underline{= 1}}$$

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c.) $$f_z(z) = \frac{\gamma^{\alpha}}{\Gamma(\alpha)} z^{\alpha-1} e^{-\gamma z} \mathbb{I}_{[0,\infty)}(z)$$

mit $$f_{\rho_r}(\rho_r) = \frac{1}{\sqrt{2\pi \rho_r \beta^2 \sigma^2}} e^{-\frac{\rho_r}{2\beta^2\sigma^2}} \mathbb{I}_{[0,\infty)}(\rho_r)$$

$$z = \rho_r$$

$$\gamma = \frac{1}{2\beta^2\sigma^2}$$

$$\alpha = \frac{1}{2}$$

$$\frac{\gamma^{\alpha}}{\Gamma(\alpha)} z^{\alpha-1} e^{-\gamma z} \mathbb{I}_{[0,\infty)}(z) = \frac{1}{\sqrt{2\beta^2\sigma^2}} \underbrace{\frac{1}{\Gamma(\frac{1}{2})}}_{=\sqrt{\pi}} \frac{1}{\sqrt{\rho_r}} e^{-\frac{\rho_r}{2\beta^2\sigma^2}} \cdot \mathbb{I}_{[0,\infty)}(z)$$

d.) $$f_{\rho_r}(\rho_r) = \sum_{k=1}^{2} f_u(T_k^{-1}(\rho_r)) \left| \frac{\partial T_k^{-1}(\rho_r)}{\partial \rho_r} \right|$$

$$= \Big[\underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\overbrace{\frac{1}{\beta}\sqrt{\rho_r}}^{u} - \mu)^2}{2\sigma^2}\right) \frac{1}{2\beta\sqrt{\rho_r}}}_{\text{Beitrag zu } i_2 \text{ für } u \geq 0}$$

$$+ \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(-\frac{1}{\beta}\sqrt{\rho_r} - \mu)^2}{2\sigma^2}\right) \cdot \frac{1}{2\beta\sqrt{\rho_r}}}_{\text{Beitrag zu } i_1 \text{ für } u<0} \Big] \mathbb{I}_{[0,\infty)}(\rho_r)$$

$$= \frac{1}{\sqrt{2\pi\beta^2 \rho_r \sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(\frac{\rho_r}{\beta^2} + a^2\right)\right)$$

$$\cdot \left\{ \frac{1}{2} \exp\left(\frac{1}{2\sigma^2} \frac{2}{\beta} \sqrt{\rho_r}\mu\right) + \frac{1}{2} \exp\left(-\frac{1}{2\sigma^2} \frac{2}{\beta} \sqrt{\rho_r}\mu\right) \right\}$$

$$= \frac{1}{\sqrt{2\pi\rho_r\beta^2\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(\frac{\rho_r}{\beta^2} + u^2\right)\right) \cdot \cosh\left(\frac{\sqrt{\rho_r}}{\sigma^2\beta}\mu\right)$$

Aufgabe 2

$$\alpha = a + i \cdot b \qquad a, b \in \mathbb{R}$$

$$z = (a+ib)\cdot(x+iy)$$

$$= ax - by + i(bx + \cancel{ae})$$

$$= v + i \cdot w$$

$$v = \mathrm{Re}(z) \qquad w = \mathrm{Im}(z)$$

$$= ax - by \qquad = bx + \cancel{ae}$$

Hilfe:

$$z = c \cdot x \qquad x \sim N(0, \sigma^2)$$

$$f_z(z) = \frac{1}{c} \cdot f_x(x = \frac{z}{c})$$

$$= \frac{1}{\sqrt{2\pi c^2 \sigma^2}} \exp\left(-\frac{z^2}{2c^2\sigma^2}\right) \sim N(0, c^2\sigma^2)$$

$$\Rightarrow \quad ax \sim N(0, a^2\tau^2)$$

$$-by \sim N(0, b^2\tau^2)$$

mit $x, y \sim N(0, \tau^2)$ i.i.d.

folgt mit Proposition 2.4.15. b.)

$$v \sim N(0, (a^2+b^2)\tau^2)$$

$$w \sim N(0, (a^2+b^2)\tau^2)$$

$$\Rightarrow v, w \sim N(0, |\alpha|^2\tau^2)$$

nach Beispiel 2.4.9.

$$x \sim N_n(\mu, \mathrm{diag}(\sigma_1^2, \dots, \sigma_n^2))$$

$$\Rightarrow x_1, \dots, x_n \quad \text{s.u. mit } X_i \sim N(\mu_i, \sigma_i^2)$$

$$\mathrm{cov}(v, w) = E(v \cdot w) - E(v) \cdot E(w)$$

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$$= E((ax-by)(bx+ay)) - 0\cdot 0$$

$$= E(abx^2) - E(aby^2) + \underbrace{\cancel{E(a^2xy)} - \cancel{E(b^2xy)}}_{x,y \text{ i.i.d. und } E(x)=E(y)=0}$$

$$= ab\tau^2 - ab\tau^2 \underline{= 0}$$

$$\Rightarrow \text{Cov}((v,w)) = \text{diag}(\sigma_1^2, \sigma_2^2)$$

$$f_{(v,w)}(v,w) = \frac{1}{\sqrt{2\pi |\alpha|^2 \tau^2}} \exp\left(-\frac{v^2}{2|\alpha|^2\tau^2}\right) \cdot \frac{1}{2\pi |\alpha|^2 \tau^2} \cdot \exp\left(-\frac{w^2}{2|\alpha|^2\tau^2}\right)$$

gesucht: $f_P(\rho)$

Transformationsfunktion

$$(P, \phi) = T(v,w) = (v^2 + w^2, \measuredangle(v,w))$$

$$\cdot T^{-1}(\rho, \phi) = (v,w) = (\sqrt{\rho}\cos(\phi), \sqrt{\rho}\sin(\phi))$$

Transformationssatz

$$f_{(\rho,\phi)}(P,\phi) = \underbrace{\left| \begin{pmatrix} \frac{\partial T^{-1}}{\partial \rho} & \frac{\partial T_1^{-1}}{\partial \phi} \\ \frac{\partial T_2^{-1}}{\partial \rho} & \frac{\partial T_2^{-1}}{\partial \phi} \end{pmatrix} \right|}_{K} \cdot f_{(v,w)}(T^{-1}(\rho,\phi))$$

$$K = \left| \begin{pmatrix} \frac{1}{2\sqrt{\rho}}\cos(\phi) & -\sqrt{\rho}\sin(\phi) \\ \frac{1}{2\sqrt{\rho}}\sin(\phi) & \sqrt{\rho}\cos(\phi) \end{pmatrix} \right|$$

$$= \frac{1}{2}\cos^2(\phi) + \frac{1}{2}\sin^2(\phi) = \frac{1}{2}$$

$$f_{(\rho,\phi)}(\rho,\phi) = \frac{1}{2}\frac{1}{\sqrt{2\pi|\alpha|^2T^2}} \exp\left(-\frac{V^2}{2|\alpha|^2T^2}\right) \cdot \frac{1}{\sqrt{2\pi|\alpha|^2T^2}} \exp\left(-\frac{w^2}{2|\alpha|^2T^2}\right)\Bigg|_{(\rho,\phi)=(V^2+w^2,\ \measuredangle(V,w))}$$

$$= \frac{1}{2}\frac{1}{2\pi|\alpha|^2T^2} \exp\left(-\frac{V^2+w^2}{2|\alpha|^2T^2}\right)\Bigg|_{(\rho,\phi)=(V^2+w^2,\ \measuredangle(V,w))}$$

$$= \frac{1}{2}\frac{1}{2\pi|\alpha|^2T^2} \exp\left(-\frac{\rho}{2|\alpha|^2T^2}\right) \mathbb{I}_{[0,\infty)}(\rho)\ \mathbb{I}_{[0,2\pi)}(\phi)$$

$$= \underbrace{\frac{1}{2\pi}\mathbb{I}_{[0,2\pi)}(\phi)}_{f_\phi(\phi)}\ \underbrace{\frac{1}{2|\alpha|^2T^2}\exp\left(-\frac{\rho}{2|\alpha|^2T^2}\right)\mathbb{I}_{[0,\infty)}(\rho)}_{f_\rho(\rho)\ \sim\ \mathrm{Exp}\left(\frac{1}{2|\alpha|^2\sigma^2}\right)}$$

b.) $$P(\rho<\lambda) = \int_0^\lambda f_\rho(\rho)\,d\rho$$

$$= \int_{-\infty}^{\lambda} \frac{1}{2|\alpha|^2\sigma^2}\exp\left(-\frac{\rho}{2|\alpha|^2\sigma^2}\right)d\rho$$

$$= 1-e^{-\frac{\lambda}{2|\alpha|^2\sigma^2}}$$