Übung 4 - MuLö

Freitag, 12. November 2010
15:00

Aufg. 1

a) $X \sim Poi(\lambda) \quad \lambda > 0$

$$P(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \quad k \in \mathbb{N}_0$$

$$E(X) = \sum_{k=0}^{\infty} k \cdot P(X=k) = \sum_{k=0}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} k \, e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{(k-1)!}$$

$$= \lambda \cdot \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!} = \lambda \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} = \lambda$$

Varianz:

$$E(X^2) = \sum_{k=0}^{\infty} k^2 \cdot P(X=k) = \sum_{k=0}^{\infty} k^2 \cdot e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} k^2 e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{(k-1)!}$$

$$= \sum_{k=1}^{\infty} (k-1) e^{-\lambda} \frac{\lambda^k}{(k-1)!} + \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{(k-1)!}$$

$$= \sum_{k=2}^{\infty} (k-1) e^{-\lambda} \frac{\lambda^k}{(k-1)!} + \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{(k-1)!}$$

$$= \lambda^2 \sum_{k=2}^{\infty} e^{-\lambda} \frac{\lambda^{k-2}}{(k-2)!} + \lambda \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda^2 \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} + \lambda \cdot \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \lambda^2 + \lambda$$

$$Var(X) = E[(X-E(X))^2]$$

$$= E(X^2) - E^2(X)$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

b) $X \sim Exp(\lambda)$, $\lambda > 0$

$$f_X(x) = \lambda \cdot e^{-\lambda x} \cdot \mathbb{1}_{[0,\infty)}(x)$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

$$= \int_0^{\infty} x \cdot \lambda \cdot e^{-\lambda x}\, dx = \frac{1}{\lambda}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^{\infty} x^2 \lambda \cdot e^{-\lambda x}\, dx$$

$$= \frac{2}{\lambda^2}$$

$$\Rightarrow Var(X) = E(X^2) - E^2(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Aufg. 2

a) X und Y seien 2 nicht spez. Zufallsvar. Zeige:

i) $Var(X) = E[(X-E(X))^2]$

$$= E(X^2) - E^2(X)$$

$$E[(X-E(X))^2] = E[X^2 - 2XE(X) + E^2(X)]$$

$$= E(X^2) - E[2XE(X)] + E(E^2(X))$$

$$= E(X^2) - 2E(X)\cdot E(X) + E^2(X)\cdot E(1)$$

$$= E(X^2) - E^2(X)$$

ii) $Cov(X,Y) = E[(X-E(X))\cdot(Y-E(Y))]$

$$= E(X\cdot Y) - E(X)\cdot E(Y)$$

$$E[(X-E(X))(Y-E(Y))]$$

$$= E(X\cdot Y - X\cdot E(Y) - Y\cdot E(X) + E(X)\cdot E(Y)]$$

$$= E(X\cdot Y) - E(Y)\cdot E(X) - E(X)\cdot E(Y) + E(X)\cdot E(Y)$$

$$= E(X\cdot Y) - E(Y)\cdot E(X)$$

b) $\underline{C} = (Cov(X_i, X_j))_{1\le i,j\le n} \in \mathbb{R}^{n\times n}$

$$\underline{C} = \begin{pmatrix} Cov(X_1, X_1) & Cov(X_1, X_2) & \cdots & Cov(X_1, X_n) \\ Cov(X_2, X_1) & & & \\ \vdots & & \ddots & \\ Cov(X_n, X_1) & \cdots & & Cov(X_n, X_n) \end{pmatrix}$$

$$= \begin{pmatrix} E[(X_1-E(X_1))^2] & \cdots & E[(X_1-E(X_1))(X_n-E(X_1))] \\ E[(X_2-E(X_2))(X_1-E(X_1))] & & \vdots \\ \vdots & \ddots & \\ E[(X_n-E(X_n))(X_1-E(X_1))] & \cdots & \end{pmatrix}$$

$\Rightarrow \underline{C} = \underline{C}^T \longrightarrow$ Symmetric

c) $\underline{X} = (X_1, X_2)'$

$$f_X(x_1,x_2) = \frac{1}{\pi\sqrt{3}} \cdot \exp\left(-\frac{2}{3}\left(x_1^2 - x_1x_2 + x_2^2\right)\right)$$

gesucht: Erwartungswertvektor $\underline{\mu}$
Cov-Matrix $\underline{C}$, mit $\det(\underline{C}) > 0$

durch Vergleich mit:

$$f_{\underline{x}}(\underline{x}) = \frac{1}{(2\pi)^{n/2} |\underline{C}|^{1/2}} \exp\left(-\frac{1}{2} (\underline{x} - \underline{\mu})' \underline{C}^{-1} (\underline{x} - \underline{\mu})\right)$$

$$\Rightarrow \pi\sqrt{3} = (2\pi) \cdot |\underline{C}|^{1/2}$$

$$\Rightarrow |\underline{C}| = \frac{3}{4}$$

Exponent:

$$-\frac{1}{2} \left[\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_1 \end{pmatrix}\right)' \cdot C^{-1} \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}\right)\right)$$

$$\stackrel{!}{=} -\frac{2}{3} (x_1^2 - x_1 x_2 + x_2^2)$$

$$\Rightarrow \underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \underline{0}$$

$$\boxed{C^{-1} = \frac{1}{\det(C)} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix}}$$

$$\Rightarrow -\frac{1}{2} (x_1 \;\; x_2) \frac{4}{3} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= -\frac{2}{3} (x_1 \;\; x_2) \begin{pmatrix} \sigma_{22} x_1 - \sigma_{12} x_2 \\ -\sigma_{12} x_1 + \sigma_{11} x_2 \end{pmatrix}$$

$$= \frac{2}{3} (\sigma_{22} x_1^2 - 2 \sigma_{12} x_1 x_2 + \sigma_{11} x_2^2)$$

$$\stackrel{!}{=} \frac{2}{3} (x_1^2 - x_1 x_2 + x_2^2)$$

$$\Rightarrow \sigma_{22} = 1 \; ; \; \sigma_{12} = \frac{1}{2} \; ; \; \sigma_{11} = 1$$

d) $\underline{X} = (X_1, X_2)$ Normalverteilt

$\to$ 1-D Normalverteilung von X_1, X_2

$\to$ es: $f_{X_1}(x_1)$ $f_{X_2}(x_2)$

$\to$ μ und σ lassen sich aus $\underline{\mu}$ und $\underline{C}$ ablesen

$\Rightarrow$ Randdichten:

$$f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} x_i^2\right)$$

$\Rightarrow$ $X_1, X_2 \sim \mathcal{N}_1(0, 1)$

↳ Standardnormalverteilung