Übung 6

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15:15

Aufg. 1

a) $f_{|h_k|}(|h_k|) = \frac{2|h_k|}{\sigma_h^2} \exp\left(-\frac{|h_k|^2}{\sigma_h^2}\right) I_{[0,\infty)}(|h_k|)$

ges: $f_{|h_k|^2}(|h_k|^2)$ $\qquad x = |h_k|$

$|h_k|^2 = T(|h_k|)$ $\qquad T(x) = x^2 = y$

$x = T^{-1}(y) = \sqrt{x}$

$\frac{\partial T(x)}{\partial x} = 2x$

$$f_{|h_k|^2}(y) = \frac{1}{\left|\frac{\partial T}{\partial x}\Big|_{x=T^{-1}(y)}\right|} \cdot f_{|h_k|}(T^{-1}(y))$$

$$= \frac{1}{\left|2x\big|_{x=\sqrt{y}}\right|} f_{|h_k|}(\sqrt{y})$$

$$= \frac{1}{2\sqrt{y}} \cdot \frac{2\sqrt{y}}{\sigma_h^2} \cdot \exp\left(-\frac{\sqrt{y}^2}{\sigma_h^2}\right) \mathbb{I}_{[0,\infty)}(\sqrt{y})$$

$$= \frac{1}{\sigma_h^2} \exp\left(-\frac{y}{\sigma_h^2}\right) \mathbb{I}_{[0,\infty)}(y)$$

$\Rightarrow$ Exp-Verteilung mit Parameter $\frac{1}{\sigma_h^2}$

b) $SNR = \frac{E[|h_k \cdot x_k|^2]}{E[|n_k|^2]}$

$$E[|n_k|^2] = E[(n_{rk} + i n_{ik})(n_{rk} - i n_{rk})]$$

$$= E[n_{rk}^2 + n_{ik}^2] = E[n_{rk}^2] + E[n_{ik}^2]$$

$$f_{n_{2K}}(z) = f_{n_{1K}}(z) = \frac{1}{\sqrt{2\sigma_n^2}} \cdot \exp\left(-\frac{z^2}{\sigma_n^2}\right)$$

$$\sim \mathcal{N}\left(0, \frac{\sigma_n^2}{2}\right)$$

$$\Rightarrow E[|n_k|^2] = \frac{\sigma_n^2}{2} + \frac{\sigma_n^2}{2} = \sigma_n^2$$

$$E[|h_K \cdot x_k|^2] = E[|h_K|^2 \cdot |x_k|^2]$$

$$\underset{\text{s.u.}}{=} \underbrace{E[|h_K|^2]}_{=\sigma_h^2} \cdot E[|x_k|^2]$$

$$\Rightarrow E[|h_K \cdot x_k|^2] = \sigma_h^2 \cdot E[|x_k|^2]$$

$$= \sigma_h^2 \cdot \sigma_x^2$$ (da mittl. Leistung von x_k σ_x^2 ist)

$$\Rightarrow SNR = \frac{\sigma_h^2 \cdot \sigma_x^2}{\sigma_n^2}$$

c) Weitere Annahme:

$|x_k|^2 = \sigma_x^2$ Sendesymbole haben konst. Leistung

gilt:

$$\frac{E[|h_k \cdot x_k|^2]}{E[|n_k|^2]} = E\left[\frac{|h_K \cdot x_k|^2}{|n_k|^2}\right] \quad ?$$

$$E\left[\frac{|h_K \cdot x_k|^2}{|n_k|^2}\right] = E[|h_K \cdot x_k|^2] \cdot E\left[\frac{1}{|n_k|^2}\right]$$

$$= \sigma_h^2 \cdot \sigma_x^2 \cdot E\left[\frac{1}{|n_k|^2}\right]$$

Analogie: $|n_k|^2$ und $|h_K|^2$

$$f_{|n_k|^2}(z) = \frac{1}{\sigma_n^2} \exp\left(-\frac{z}{\sigma_n^2}\right) \mathbb{1}_{[0,\infty)}(z)$$

$$= \int^{\infty}$$

$$E\left[\frac{1}{|h_k|^2}\right] = \int_{-\infty}^{\infty} \frac{1}{z} f_{|h_k|^2}(z)\, dz$$

$$= \int_0^{\infty} \frac{1}{z} \cdot \frac{1}{\sigma_h^2} \exp\left(-\frac{z}{\sigma_h^2}\right) dz \qquad \left[\; y = \frac{z}{\sigma_h^2},\quad dz = \sigma_h^2\, dy \;\right]$$

$$= \int_0^{\infty} \frac{1}{y \cancel{\sigma_h^2}} \frac{1}{\sigma_h^2} \exp(-y) \cdot \cancel{\sigma_h^2}\, dy$$

$$= \frac{1}{\sigma_h^2} \int_0^{\infty} \frac{1}{y} \cdot \exp(-y)\, dy$$

$$= \frac{1}{\sigma_h^2} \lim_{x \to 0} \int_0^{\infty} \frac{1}{y} \exp(-y)\, dy$$

$$= -\frac{1}{\sigma_h^2} \lim_{x \to 0} Ei(-x) = \infty$$

$$\left[\; \underline{Def.}: \quad Ei(z) = -\int_z^{\infty} \frac{e^{-t}}{t}\, dt \;\right]$$

$$\rightarrow E\left[\frac{|h_k \cdot x_k|^2}{|h_k|^2}\right] = \infty \neq \frac{E[|h_k \cdot x_k|^2]}{E[|h_k|^2]} = \frac{\sigma_h^2 \cdot \sigma_x^2}{\sigma_h^2}$$

d) $h_k = h_{Rk} + i\, h_{Ik}$

$E[h_{Rk} \cdot h_{Rk+1}] = E[h_{Ik} \cdot h_{Ik+1}] = c$

$|x_k| = \sigma_x$

Eingangssymbole x_k sollen unabh. Phasen haben

$\underline{Y} = \begin{pmatrix} Y_{R1} \\ Y_{I1} \\ Y_{R2} \\ Y_{I2} \end{pmatrix}$ sind Elem. von $\underline{Y}$ $\underline{\underline{\text{gem.}}}$ normal Verteilt

$$y_k = x_k \cdot h_k + n_k$$

$$= (x_{Rk} + i \cdot x_{ik}) \cdot (h_{Rk} + i h_{ik}) + n_{Rk} + i n_{ik}$$

$$= x_{Rk} \cdot h_{Rk} + i \cdot x_{Rk} \cdot h_{ik} + i\, x_{ik} \cdot h_{Rk} - x_{ik} h_{ik} + n_{Rk} + i n_{ik}$$

$$= \underbrace{x_{Rk} \cdot h_{Rk} - x_{ik} \cdot h_{ik} + n_{Rk}}_{Re(y_k) = y_{Rk}} + i \underbrace{(x_{Rk} h_{ik} + x_{ik} h_{Rk} + n_{ik})}_{Im(y_k) = y_{ik}}$$

$$\underline{Y} = \begin{pmatrix} y_{R1} \\ y_{i1} \\ y_{R2} \\ y_{i2} \end{pmatrix} = \begin{pmatrix} x_{R1} h_{R1} - x_{i1} h_{i1} + n_{R1} \\ x_{i1} h_{R1} + x_{R1} h_{i1} + n_{i1} \\ x_{R2} h_{R2} - x_{i2} h_{i2} + n_{R2} \\ x_{i2} h_{R2} + x_{R2} h_{i2} + n_{i2} \end{pmatrix}$$

x_k und h_k sind s.a.

$$E[x_k] = E[h_k] = E[n_k] = 0$$

$$\Rightarrow E[\underline{Y}] = 0$$

$$Cov[\underline{Y}] = E[\underline{Y}\,\underline{Y}^T] = E\left[\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}\right]$$

$$E[a_{11}] = E[(x_{R1} \cdot h_{R1} - x_{i1} h_{i1} + n_{R1})^2]$$

$$= E[x_{R1}^2 h_{R1}^2 - x_{R1} \cancel{h_{R1} x_{i1}} h_{i1} + x_{R1} \cancel{h_{R1}} n_{R1} - x_{i1} \cancel{h_{i1} x_{R1}} h_{R1}$$

$$+ x_{i1}^2 h_{i1}^2 - x_{i1} h_{i1} \cancel{n_{R1}} + n_{R1} \cancel{x_{R1}} h_{R1} - n_{R1} \cancel{x_{i1}} h_{R1} + n_{R1}^2]$$

$[h_{Rk}$ s.a. von $h_{ik}]$

$$= E[x_{R1}^2 h_{R1}^2 + x_{i1}^2 h_{i1}^2 + n_{R1}^2]$$

$$= E[x_{R1}^2] \underbrace{E[h_{R1}^2]}_{\sigma_h^2/2} + E[x_{i1}^2] \underbrace{E[h_{i1}^2]}_{\sigma_h^2/2} + \underbrace{E[n_{R1}^2]}_{\sigma_n^2/2}$$

da, $h_{R1} \sim h_{i1} \sim \mathcal{N}(0, \sigma_h^2/2)$

$$= \left(E[x_{R1}^2] + E[x_{I1}^2]\right) \cdot \sigma_h^2/2 \quad + \quad \sigma_n^2/2$$

$$= E[|x_1|^2] \cdot \sigma_h^2/2 \quad + \quad \sigma_n^2/2$$

$$= \sigma_x^2 \cdot \sigma_h^2/2 \quad + \quad \sigma_n^2/2$$

$$E[a_{12}] = [(x_{R1} h_{R1} - x_{I1} h_{I1} + n_{R1})(x_{I1} h_{R1} + x_{R1} h_{I1} + n_{I1})]$$

$$= E[x_{R1} x_{I1} h_{R1}^2 - x_{I1} x_{R1} h_{I1}^2]$$

$$= \sigma_h^2 \cdot E[x_{R1} x_{I1} - x_{I1} x_{R1}] = 0$$

$$E[a_{13}] = E[(x_{R1} h_{R1} - x_{I1} h_{I1} + n_{R1})(x_{R2} h_{R2} - x_{I2} h_{I2} + n_{R2})]$$

$$= E[x_{R1} h_{R1} \cdot x_{R2} h_{R2} + x_{I1} h_{I1} x_{I2} h_{I2}]$$

$E[x_{R1} \cdot x_{R2}] = 0$

$$= 0$$

$$E[a_{14}] = E[(x_{R1} h_{R1} - x_{I1} h_{I1} + n_{R1})(x_{I2} h_{R2} + x_{R2} h_{I2} + n_{I2})]$$

$$= 0$$

$$E[a_{21}] = E[a_{12}] = 0$$

$$E[a_{22}] = E[(x_{I1} h_{R1} + x_{R1} h_{I1} + n_{I1})(x_{I1} h_{R1} + x_{R1} h_{I1} + n_{I1})]$$

$$= E[x_{R1}^2 h_{R1}^2 + x_{R1}^2 h_{I1}^2 + n_{I1}^2]$$

$$= E[x_{I1}^2]\, E[h_{R1}^2] + E[x_{R1}^2]\, E[h_{I1}^2] + E[n_{I1}^2]$$

$$= \frac{\sigma_h^2}{2} E[|x_1|^2] + \frac{\sigma_n^2}{2}$$

$$= \frac{\sigma_h^2}{2} \sigma_x^2 + \frac{\sigma_n^2}{2}$$

$$E[a_{23}] = E[(x_{I1} h_{R1} + x_{R1} h_{I1} + n_{I1})(x_{R2} h_{R2} - x_{I2} h_{I2} + n_{R2})]$$

$$= 0$$

$$E[a_{24}] = E[(x_{I1} h_{R1} + x_{R1} h_{I1} + n_{I1})(x_{I2} h_{R2} + x_{R2} h_{I2} + n_{I2})]$$

$$= 0$$

$$E[a_{31}] = E[a_{13}] =$$

$E[a_{32}] = E[a_{23}]$

$E[a_{33}] = E[a_{11}]$

$E[a_{34}] = E[(x_{22}h_{22} - x_{12}h_{12} + n_{22})(x_{12}h_{22} + x_{22}h_{12} + n_{12})]$

$= E[x_{22}x_{12}h_{22}^2 - x_{12}x_{22}h_{12}^2]$

$= \frac{\sigma_h^2}{2} E[x_{22}x_{12} - x_{12}x_{22}]$

$= 0$

$E[a_{41}] = E[a_{14}]$

$E[a_{42}] = E[a_{24}]$

$E[a_{43}] = E[a_{34}]$

$E[a_{44}] = E[a_{11}]$

$$\Rightarrow Cov[\underline{Y}] = \left(\frac{\sigma_x^2 \cdot \sigma_h^2}{2} + \frac{\sigma_n^2}{2}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Wenn $\underline{Y}$ gem. normalverteilt wäre, dann würde eine diag. Kovarianzmatrix implizieren, dass y_1 und y_2 s.u. wären

$y_k = x_k \cdot h_k + n_k$

$|x_k| = \sigma_x$

Annahme: $\sigma_n^2 = 0$

$E[h_{2k} h_{2k+1}] = E[h_{1k} h_{1k+1}] = C = \frac{\sigma_h^2}{2}$

$\Rightarrow$ Kanal sich zwischen Zeitpunkt 1 und 2 nicht ändert

für diesen Spezialfall $Cov[|y_1|^2 / |y_2|^2]$

$E[|y_k|^2] = E[|x_k h_k|^2] = E[|x_k|^2] E[|h_k|^2]$

$= \sigma_x^2 \cdot \sigma_h^2$

$$\hookrightarrow \mathrm{Cov}[|y_1|^2|y_2|^2] = E[|y_1|^2|y_2|^2] - E[|y_1|^2]\,E[|y_2|]$$

$$= E[|x_1|^2|h_1|^2|x_2|^2|h_2|^2] - \sigma_x^4\sigma_h^4$$

$$= E[|x_1|^2]\,E[x_2^2]\,E[|h_1|^2|h_2|^2] - \sigma_x^4\sigma_h^4$$

$$= \sigma_x^4 \cdot E[|h_1|^2|h_2|^2] - \sigma_x^4\sigma_h^4$$

$$= \sigma_x^4 \cdot E[(h_{R1}^2 + h_{I1}^2)(h_{R2}^2 + h_{I2}^2)] - \sigma_x^4\sigma_h^4$$

$$= \sigma_x^4 \cdot E[h_{R1}^2 h_{R2}^2 + h_{R1}^2 h_{I2}^2 + h_{I1}^2 h_{R2}^2 + h_{I1}^2 h_{I2}^2] - \sigma_x^4\sigma_h^4$$

$$= \sigma_x^4\, E[h_{R1}^4 + h_{R1}^2 h_{I2}^2 + h_{I1}^2 h_{R1}^2 + h_{I1}^4] - \sigma_x^4\sigma_h^4$$

$$= \sigma_x^4 \left[3\left(\frac{\sigma_h^2}{2}\right) + \frac{\sigma_h^2}{2}\cdot\frac{\sigma_h^2}{2} + \frac{\sigma_h^2}{2}\,\frac{\sigma_h^2}{2} + 3\left(\frac{\sigma_h^2}{2}\right)^2\right] - \sigma_x^4\sigma_h^4$$

$$= \sigma_x^4 \cdot \sigma_h^4 \neq 0$$

$\Rightarrow$ $|y_1|^2$ und $|y_2|^2$ im Allgem. nicht unabh.

$\Rightarrow$ $\underline{y}$ kann nicht gem. normalverteilt sein