GGÜ 3

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09:58

Aufgabe 5

- positive z-Richtung
- verlustbehaltetes Medium

$$\underline{k} = k' - j k''$$

$$\vec{E}(\vec{r},t) = \mathrm{Re}\{ E_0 e^{-j\underline{k}z} e^{j\omega t} \vec{e}_x$$
$$= E_0 \vec{e}_x \mathrm{Re}\{ e^{j(\omega t - \underline{k} z)}\}$$

$$\vec{H}(\vec{r},t) = \mathrm{Re}\{ \underline{\vec{H}} e^{-j\underline{k}z} e^{j\omega t}\}$$

a) $\Delta \mathrm{Re}\{\underline{\vec{E}}(\vec{r},t)\} - \mu\sigma \frac{\partial}{\partial t} \mathrm{Re}\{\underline{\vec{E}}(\vec{r},t)$
$- \varepsilon\mu \frac{\partial^2}{\partial t^2} \mathrm{Re}\{\underline{\vec{E}}(\vec{r},t)\} = \vec{0}$

mit $\underline{\vec{E}}(\vec{r},t) = \underline{\vec{E}}(\vec{r}) \cdot e^{j\omega t} = \underline{\vec{E}} e^{j\omega t}$

nun Substitution von $t = t' - \frac{T}{4}$

$\Rightarrow \Delta \mathrm{Re}\{\underline{\vec{E}}(\vec{r}, t' - \frac{T}{4})\} - \mu\sigma \frac{\partial}{\partial t} \ldots$

$\Delta \mathrm{Im}\{\underline{\vec{E}}(\vec{r},t')\} - \mu\sigma \ldots$

$\Delta \mathrm{Re}\{\underline{\vec{E}}(\vec{r},t)\} + j\Delta \mathrm{Im}\{\underline{\vec{E}}(\vec{r},t)\} \ldots$

$$\Delta \underline{\vec{E}}(\vec{r},t) - \mu\sigma \frac{\partial}{\partial t} \underline{\vec{E}}(\vec{r},t) - \varepsilon\mu \frac{\partial^2}{\partial t^2} \underline{\vec{E}}(\vec{r},t) = \vec{0}$$

a) Berechnung der zeitl. Ableitung

$$\Delta \underline{\vec{E}}(\vec{r},t) - \mu j\omega\sigma \underline{\vec{E}}(\vec{r},t) + \omega^2 \varepsilon\mu \underline{\vec{E}}(\vec{r},t) = \vec{0}$$

mit $\omega^2 \cdot \varepsilon\mu - \mu j\omega\sigma = \omega^2\mu\left(\varepsilon - j\frac{\sigma}{\omega}\right)$

$$\Delta \underline{\vec{E}} + \omega^2 \underline{\varepsilon}\mu \underline{\vec{E}} = 0$$

$$\underbrace{\frac{\partial^2}{\partial x^2} \underline{\vec{E}}}_{0} + \underbrace{\frac{\partial^2}{\partial y^2} \underline{\vec{E}}}_{0} + \frac{\partial^2}{\partial z^2} \underline{\vec{E}} + \omega^2 \underline{\varepsilon}\mu \underline{\vec{E}} = 0$$

$$\frac{\partial^2}{\partial z^2} \underline{\vec{E}} = -\underline{k}^2 \underline{\vec{E}} = -\omega^2 \cdot \underline{\varepsilon}\mu \underline{\vec{E}}$$

$$k^2 = \omega^2 \underline{\varepsilon}\mu \qquad \underline{\varepsilon} = \varepsilon - j\frac{\sigma}{\omega}$$

b) allg. für TEM

$$\vec{H} = \frac{1}{Z}(\vec{n} \times \underline{\vec{E}})$$

$$\text{rot}\,\vec{E} = -\mu\frac{\partial \vec{H}}{\partial t} \Rightarrow \text{rot}\,\underline{\vec{E}} = -j\omega\mu \underline{\vec{H}}$$

$$\underline{\vec{H}} = +\frac{j}{\omega\mu}\,\text{rot}\,\vec{E}$$

$$= \frac{j}{\omega\mu}\Big[\Big(\frac{\partial}{\partial z} E_x \vec{e}_y - \underbrace{\frac{\partial}{\partial y} E_x \vec{e}_z}_{=0}\Big) + \Big(\underbrace{\frac{\partial}{\partial x} E_y \vec{e}_z}_{=0} - \frac{\partial}{\partial z} E_y \vec{e}_x\Big) + \Big(\underbrace{\frac{\partial}{\partial y} E_z \vec{e}_x}_{=0} - \underbrace{\frac{\partial}{\partial x} E_z \vec{e}_y}_{=0}\Big)\Big]$$

$$= \frac{j}{\omega\mu}\frac{\partial}{\partial z}(\underline{E}_x \vec{e}_y - \underline{E}_y \vec{e}_x) \text{ mit } \underline{\vec{E}} = \underline{\vec{A}}_0\, e^{-j\underline{k}z}$$

$$H = \frac{j}{\omega\mu}(-j\underline{k}) \cdot \underbrace{(\underline{E}_x \cdot \vec{e}_y - \underline{E}_y \vec{e}_x)}_{-\underline{\vec{E}}_x \vec{e}_z = +\vec{e}_z \times \underline{\vec{E}}}$$

$$= \frac{\underline{k}}{\omega\mu} \left(\vec{e}_z \times \underline{\vec{E}}\right)$$

$$\frac{\underline{k}}{\omega\mu} = \frac{\sqrt{\omega^2 \underline{\varepsilon} \mu}}{\omega\mu} = \sqrt{\frac{\underline{\varepsilon}}{\mu}} = \frac{1}{\underline{Z}}$$

$$\underline{\vec{H}} = \frac{1}{\underline{Z}} \underline{E} \, e^{j\underline{k}z} \left(\vec{e}_z \times \vec{e}_x\right)$$

$$= \frac{1}{\underline{Z}} E_0 \vec{e}_y e^{-j\underline{k}z}$$

$\Rightarrow$ Phasenverschiebung zw E und H

c) $$\overline{\vec{S}} = \frac{1}{2} \operatorname{Re}\{\underline{\vec{S}}\} = \frac{1}{2} \operatorname{Re}\{\underline{\vec{E}} \times \underline{\vec{H}}^*\}$$

$$\vec{S} = \operatorname{Re}\{\underline{\vec{E}}(\vec{r},t)\} \times \operatorname{Re}\{\underline{\vec{H}}(\vec{r},t)\}$$

$$\underline{\vec{S}} = \underline{\vec{E}} \times \underline{\vec{H}}^*$$

$$\overline{\vec{S}} = \frac{1}{2} \operatorname{Re}\left\{ E_0 \vec{e}_x e^{-j\underline{k}z} \times \frac{E_0}{\underline{Z}^*} \vec{e}_y e^{-j\underline{k}^* z} \right\}$$

$$= \frac{1}{2} \operatorname{Re}\left\{ \frac{E_0^2}{\underline{Z}^*} \vec{e}_z e^{-j k' z - k'' z} \cdot e^{j k' z - k'' z} \right\}$$

$$= \frac{1}{2} E_0^2 \vec{e}_z e^{-2k''z} \frac{k'}{2\mu\omega}$$

d) $$\underline{k} = \sqrt{\omega^2 \mu \underline{\varepsilon}} = \omega\sqrt{\mu\left(\varepsilon - j\frac{\sigma}{\omega}\right)}$$

$$= \omega\sqrt{\mu} \sqrt{-j\frac{\sigma}{\omega}} \sqrt{1 - \frac{\omega\varepsilon}{j\sigma}} \qquad \text{mit } \sigma \gg \omega\varepsilon$$

$$\approx \sqrt{\frac{\omega\mu\sigma}{2}} (1-j) \Big(\underbrace{1 - \frac{\omega\varepsilon}{j2\sigma}}_{\approx 1}\Big) = \frac{1-j}{\sqrt{2}}$$

$$\underline{k} \approx \sqrt{\frac{\omega\mu\sigma}{2}} (1-j)$$

e^{-jkz} | Re

$$-jkz = -\sqrt{\frac{\omega\mu\sigma}{2}}\,(1+j) = \lambda$$

$$k' = \sqrt{\frac{\omega\mu\sigma}{2}} \qquad k'' = +\sqrt{\frac{\omega\mu\sigma}{2}}$$

$$\frac{1}{k''} = \sqrt{\frac{2}{\omega\mu\sigma}} = a$$

$$v = \frac{\omega}{k'}$$

$$\vec{E}(\vec{r},t) = E_0\, e^{j(k'z - jk''z + \omega t)}$$

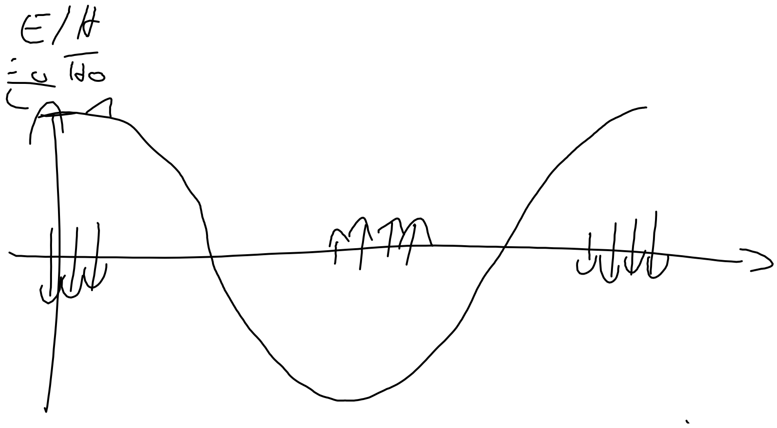
$$\vec{E}(\vec{r},t) = E_0\, e^{j(-k'z + jk''z + \omega t)} + \frac{e^{j(-k'z + jk''z + \omega t)^*}}{2}$$

$$= \frac{E_0}{2}\, e^{-jk''z} \underbrace{\left(e^{j(\omega t - k'z)} + e^{-j(\omega t - k'z)}\right)}_{2\cos(\omega t - k'z)}$$

$$\omega t - k'z = \text{const} \quad \Big| \frac{\partial}{\partial t}$$

$$\omega - k' \cdot \underbrace{\frac{\partial z}{\partial t}}_{v} = 0 \qquad v = \frac{\omega}{k'} = \frac{\omega}{\sqrt{\frac{\omega\mu\sigma}{2}}} = \sqrt{\frac{2\omega}{\mu\sigma}}$$

$$\frac{v}{v_0} = \frac{\sqrt{\frac{2\omega}{\mu\sigma}}}{\frac{1}{\sqrt{\mu\varepsilon}}} = \sqrt{\frac{2\omega\varepsilon}{\sigma}} \ll 1$$



$$\text{rot}\,\vec{H} = \vec{J} = \sigma\vec{E} = \sigma E_x \vec{e}_x$$

$$\text{rot}\,\vec{H} = -\frac{\partial}{\partial z} H_y \vec{e}_x$$

$$\frac{\partial}{\partial z} H_y \overset{!}{=} -\sigma E_x$$

A6

a) $\Delta \underline{\vec{J}} + \underline{k}^2 \underline{J} = 0$

mit $\underline{k} = \frac{1-j}{a}$; $a = \sqrt{\frac{2}{\omega\mu\sigma}}$

$$\text{rot}\,\underline{\vec{E}} = -j\omega\mu\underline{\vec{H}}$$

$$\text{rot}\,\underline{\vec{H}} = \sigma\underline{\vec{E}} + j\omega\varepsilon\underline{\vec{E}} \qquad \omega\tau = \omega\frac{\varepsilon}{\sigma} \ll 1$$

$$= \sigma\underline{\vec{E}}(1 + j\omega\tau) \approx \sigma\underline{\vec{E}}$$

$\text{div rot}\,\underline{\vec{H}} = 0 \Rightarrow \text{div}\,\underline{\vec{E}} = 0$ gilt immer im Frequenzbereich (HEZ) wenn $\varepsilon, \sigma = \text{const.}$

$$\text{rot rot}\,\vec{E} = -j\omega\mu\,\text{rot}\,\underline{\vec{H}} = -j\omega\mu\sigma\underline{\vec{E}}$$

$$\Rightarrow \text{rot rot}\,\underline{\vec{E}} = \nabla\times(\nabla\times\underline{\vec{E}}) = \nabla(\nabla\underline{\vec{E}}) - \underline{\vec{E}}(\nabla 0)$$

$$= \nabla(\underbrace{\nabla\underline{\vec{E}}}_{0}) - \underbrace{\nabla^2\underline{\vec{E}}}_{\Delta\underline{\vec{E}}}$$

$$-\Delta\underline{\vec{E}} = -j\omega\mu\sigma\underline{\vec{E}}$$

$$\Rightarrow \Delta(\sigma\underline{\vec{E}}) - j\omega\mu\sigma(\sigma\underline{\vec{E}}) = 0 \qquad \sigma\underline{\vec{E}} = \underline{\vec{J}}$$

$$\Delta \underline{\vec{J}} + \underline{k}^2 \underline{\vec{J}} = 0$$

$$\underline{k}^2 = -j\omega\mu\sigma = \frac{(1-j)^2}{2}\omega\mu\sigma$$

$$\Rightarrow \underline{k} = (1-j)\sqrt{\frac{\omega\mu\sigma}{2}} = \frac{1-j}{a} \qquad a = \sqrt{\frac{2}{\omega\mu\sigma}}$$

b) $\underline{\vec{J}} = \underline{J}(\rho)\,\vec{e}_z$

$$(\Delta \underline{\vec{J}})_\rho = 0 \qquad (\Delta \underline{\vec{J}})_\varphi = 0$$

$$(\Delta \underline{\vec{J}})_z = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{d\underline{J}_z}{d\rho}\right)$$

$$\Rightarrow \frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{d\underline{J}_z}{d\rho}\right) + \underline{k}^2 \underline{J}_z = 0$$

Bessel-DGL
0. Ordnung

$$\left(\underline{k}^2 - \frac{n^2}{\rho^2}\right) \quad n=0 \qquad \text{f.}\ \underline{k} \neq 0$$

c) $\underline{J}_z(\rho)$ soll gelöst werden

$$\underline{J}_z(\rho) = \underline{A}\,\underline{J}_0(\underline{k}\rho) + \underline{B}\,N_0(\underline{k}\rho)$$

$\underline{B} \to 0$ da $\underline{J}_z$ endlich bleiben muss für $\rho \to 0$

$\Rightarrow \underline{B} = 0$

$$\int_{LQ} \underline{J}_z(\rho)\,dF \overset{!}{=} \underline{I} \qquad \text{L.Q.} = \text{Leiterquerschnitt}$$

$$dF = \rho\, d\varphi\, d\rho \quad ; \text{L.Q.}\ 0 \le \rho \le b, \quad 0 \le \varphi \le 2\pi$$

$$\underline{I} = \underline{A}\int_0^{2\pi} d\varphi \int_0^b d\rho\cdot\rho\; J_0(\underline{k}\rho)$$

$$\underline{k}\,\rho = u \Rightarrow du = \underline{k}\,d\rho \;;\quad \rho = 0 \Rightarrow u = 0$$
$$\rho = b \Rightarrow u = \underline{k}\,b$$

$$\underline{I} = \underline{A}\,2\pi \frac{1}{\underline{k}^2} \int_0^{\underline{k}b} du\; u \cdot \underline{J}_0(u)$$

Formelsammlung: $u\,\underline{J}_0(u) = \frac{d}{du}(u\,J_1(u))$; $J_1(0) = 0$

$$\Rightarrow \underline{I} = \frac{2\pi \underline{A}}{\underline{k}^2} \int_0^{kb} \frac{d}{du}(u\,J_1(u))\,du = \frac{2\pi \underline{A}}{\underline{k}^2}\, u\,J_1(u)\Big|_0^{kb}$$

$$= \frac{2\pi \underline{A}}{\underline{k}^2}\, \underline{k}\, b \cdot J_1(\underline{k}\,b)$$

konstant

$$\Rightarrow \underline{A} = \frac{\underline{k}\,\underline{I}}{2\pi b\, J_1(\underline{k}\,b)}$$

$$\underline{\vec{J}} = \vec{e}_z\, \frac{I(1-j)}{2\pi a b}\, \frac{J_0(\underline{k}\,\rho)}{J_1(\underline{k}\,b)} \qquad \underline{k} = \frac{1-j}{a}$$

d) $\underline{\vec{H}} = \frac{1}{\omega\mu} \operatorname{rot} \underline{\vec{E}} = \frac{j}{\omega\mu\sigma} \operatorname{rot} \underline{\vec{J}} = \frac{j a^2}{2} \operatorname{rot} \underline{\vec{J}}$

$$\operatorname{rot} \underline{\vec{J}} = \operatorname{rot} (\underline{J}_z(\rho)\,\vec{e}_z)$$
$$= -\frac{\partial \underline{J}_z}{\partial \rho}\, \vec{e}_\varphi$$

$$\underline{\vec{H}} = \underline{H}_\varphi(\rho)\,\vec{e}_\varphi$$

$$\underline{H}_\varphi = -\frac{j a^2}{2} \frac{\partial J_z}{\partial \rho} = -\frac{j a^2}{2} \frac{I(1-j)}{\pi a b J_1(\underline{k}b)} \frac{d}{d\rho} J_0(\underline{k}\,\rho)$$

$$\frac{d}{d\rho} J_0(\underline{k}\rho) = \underline{k}\frac{d}{du} J_0(u)$$

Formelsammlung: $\frac{d J_0(u)}{du} = -J_1(u)$

$$\Rightarrow \frac{d}{d\rho} J_0(\underline{k}\rho) = -\underline{k} J_1(k\rho) = -\frac{1-j}{a} J_1(\underline{k}\rho)$$

$$\underline{H}_\varphi = \frac{I}{2\pi b}\frac{J_1(\underline{k}\rho)}{J_1(\underline{k}b)} \qquad \underline{k} = \frac{1-j}{a}$$

für $\rho = b \Rightarrow H_\rho$ wie erwartet

e) schwacher Skineffekt

Eindringtiefe $a \gg$ Radius b $\quad \rho \leq b \ll a$

$$\frac{\rho}{a} \ll 1$$

$$|\underline{k}\rho| = \frac{\sqrt{2}}{a}\rho \ll 1$$

$$J_0(\beta) \approx 1 - \frac{\beta^2}{2^2}$$

$$J_1(\beta) = \frac{\beta}{2} - \frac{1}{2}\left(\frac{\beta}{2}\right)^3 \qquad \text{f. } |\beta| \ll 1$$

$$J_0(\underline{k}\rho) \approx 1 - \left(\frac{\underline{k}\rho}{2}\right)^2 = 1 - \frac{\rho^2}{4a^2}\underbrace{(1-j)^2}_{-2j}$$

$$= 1 + \frac{j\rho^2}{2a^2}$$

$$J_1(\underline{k}\rho) \approx \frac{\underline{k}\rho}{2} - \frac{1}{2}\left(\frac{\underline{k}\rho}{2}\right)^3 = \frac{\underline{k}\rho}{2}\left(1 - \frac{1}{2}\left(\frac{\underline{k}\rho}{2}\right)^2\right)$$

$$= \frac{1-j}{2}\frac{\rho}{a}\left(1 + j\frac{\rho^2}{4a^2}\right) \qquad \frac{1}{1+j\bar{a}} \approx 1 - j\bar{a} \text{ f. kl. } a$$

$$J_0(\underline{k}\rho) \approx \frac{2a}{}\left(1 + j\frac{\rho^2}{}\right)\Big/\left(1 - j\frac{b^2}{}\right)$$

$$\frac{J_0(\underline{k}\rho)}{J_1(\underline{k}b)} \approx \frac{2a}{b(1-j)}\left(1+\frac{j\rho^2}{2a^2}\right)\left(1-j\frac{b^2}{4a^2}\right)$$

$$\approx \frac{2a}{b(1-j)}\left[1+\frac{j\rho^2}{2a^2}-j\frac{b^2}{4a^2}\right]$$

$$\frac{J_1(\underline{k}\rho)}{J_1(\underline{k}b)} \approx \frac{\rho}{b}\left(1+j\frac{\rho^2}{4a^2}\right)\left(1-\frac{b^2}{4a^2}\right) \approx \frac{\rho}{b}\left[1-j\frac{b^2}{4a^2}+\frac{\rho^2}{4a^2}\right]$$

$$\underline{\vec{J}} \approx \vec{e}_z \frac{I(1-j)}{2\pi ab}\frac{2a}{b(1-j)}\left[1+j\frac{1}{2}\left(\frac{\rho}{a}\right)^2-j\frac{b^2}{4a^2}\right]$$

$$\underline{\vec{H}} = \vec{e}_\varphi \cdot \frac{I}{2\pi b^2}\rho \cdot \left[1+\frac{j}{4}\left(\left(\frac{\rho}{a}\right)^2-\left(\frac{b}{a}\right)^2\right)\right]$$

Re$(\underline{J})$