V 3

Freitag, 12. November 2010
10:02

10.3 Wellenausbreitung in unbegrenzten, homogenen und isotropen Medien

$$\underline{\varepsilon, \mu, \sigma}$$

$$\Delta \vec{E} - \mu\sigma \dot{\vec{E}} - \mu\varepsilon \ddot{\vec{E}} = \vec{0}$$ Telegraphengleichung

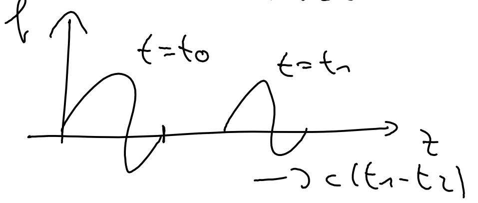
Wellengleichung: $c = 1/\sqrt{\varepsilon\mu}$

$$\Delta \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0}$$

$$\vec{E} = E_x(z,t)\vec{e}_x$$

$$\frac{\partial^2 E_x}{\partial z} - \frac{\partial^2 E_x}{\partial (ct)^2} = 0$$

$$E_x(z,t) = f(\underbrace{t-z}_{\text{Phase}}) + g(ct+z)$$



$$ct - z = \text{const}$$

$$z(t) = ct - \text{const}$$

$$\frac{dz}{dt} = c$$ Phasengeschwindigkeit

Ebene, linear polarisierte Welle

ε, μ $\quad c = \frac{1}{\sqrt{\varepsilon\mu}}$

$$\vec{E} = E_x(ct-z)\vec{e}_x \quad : \frac{\partial \vec{E}}{\partial x} = \frac{\partial \vec{E}}{\partial y} = \vec{0}$$

$$\text{rot}\, \vec{E} = \nabla \times (E_x \vec{e}_x) = \frac{\partial E_x}{\partial z}\vec{e}_y = -\mu \dot{\vec{H}} = -\mu \frac{\partial H_y}{\partial t}\vec{e}_y$$

$$\vec{H}(z,t) = H_y(ct-z)\vec{e}_y$$

$$\frac{\partial E_x}{\partial z} = -\mu c \frac{\partial H_y}{\partial ct} = \mu c \frac{\partial H_y}{\partial z} = Z \frac{\partial H_y}{\partial z} \quad ; \; Z = \sqrt{\frac{\mu}{\varepsilon}} \approx 370\,\Omega \text{ (Vakuum)}$$

$$\frac{\partial E_x}{\partial z} = Z \frac{\partial H_y}{\partial z}$$

$$\underline{E_x = Z H_y}$$

$$E_x = Z H_y$$

$$\vec{E} = E_{y}(ct - z)\,\vec{e}_y \Rightarrow E_y = -Z H_x$$

$$\vec{E}\,\vec{H} = E_x H_x + E_y H_y = Z H_y H_x + Z H_x H_y = 0$$

$$\Rightarrow \vec{E} \perp \vec{H}$$

$$\vec{E} \perp \vec{e}_z \,, \quad \vec{H} \perp \vec{e}_z$$

$$\operatorname{div} \vec{E} = 0 = \underbrace{\cancel{\frac{\partial E_x}{\partial x}}}_{=0} + \underbrace{\cancel{\frac{\partial E_y}{\partial y}}}_{=0} + \frac{\partial E_z}{\partial z} = 0$$

$$\Rightarrow E_z = 0$$

$$\operatorname{div} \vec{H} = 0 \Rightarrow H_z = 0$$

$$\vec{H} = \frac{1}{Z}(\vec{e}_z \times \vec{E})$$

<u>TEM-Welle</u> : transversal-EM-Welle

$|\vec{n}| = 1$ $\quad$ $\vec{n}$ Richtung

$\vec{E} = \vec{E}(ct - \vec{n}\vec{r})$ $\quad$ Welle in positive $\vec{n}$ Richtung

$$\vec{H} = \vec{H}(ct - \vec{n}\vec{r})$$

$$\vec{E}\vec{H} = 0 \Rightarrow \vec{E} \perp \vec{H}$$

$$\vec{E}\vec{n} = 0 \; ; \; \vec{H}\vec{n} = 0$$

$$\vec{E} \perp \vec{n} \; ; \; \vec{H} \perp \vec{n}$$

$$|\vec{E}| = Z|\vec{H}|$$

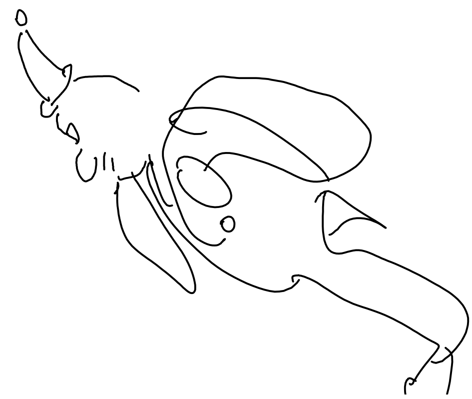
$$\vec{H} = \frac{1}{Z}(\vec{n} \times \vec{E})$$

E ebene Welle in z-Richtung

HEZ, schwache Verluste

$$\vec{E} = E_x(z,t)\,\vec{e}_x$$

$$\frac{\partial^2 \underline{E}_x}{\partial z^2} + \underline{k}^2 \underline{E}_x = 0 \qquad \underline{E}_x = \underline{A}_1 e^{-j\underline{k}z} + \underline{A}_2 e^{j\underline{k}z}$$



∂z^2

$\underline{k}^2 = \omega^2 \varepsilon\mu - j\omega\sigma\mu$

$\underline{k} = \sqrt{\omega^2\varepsilon\mu - j\omega\sigma\mu} = \omega\sqrt{\varepsilon\mu}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}$

schwache Verluste: $\sigma \ll \varepsilon\omega$ $\qquad \sqrt{1-x} = 1 - \frac{x}{2}$

$\underline{k} = k\left(1 - j\frac{\sigma}{2\omega\varepsilon}\right) = k' - jk'' \; ; \; j\underline{k} = \frac{k\sigma}{2\omega\varepsilon} + jk = \alpha + j\beta$

Re $\qquad$ Im

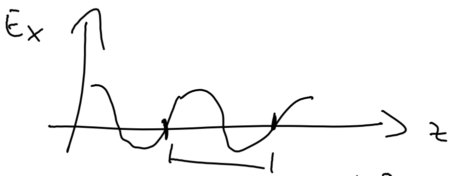
$E_x(z,t) = \mathrm{Re}\left\{ \underline{A}_1 e^{-j\underline{k}z + j\omega t} + \underline{A}_2 e^{jkz + j\omega t} \right\} \qquad \underline{A}_i = \hat{A}_i e^{j\varphi_{xi}}$

$= \hat{A}_1 \cos(\omega t - kz + \varphi_{x1})\, e^{-\alpha z} + \hat{A}_2 \cos(\omega t + kz + \varphi_{x2})\, e^{+\alpha z}$

Bsp: $\hat{A}_2 = 0 \qquad \sigma = 0 \Rightarrow \alpha = 0$

ungedämpfte harmonische Welle

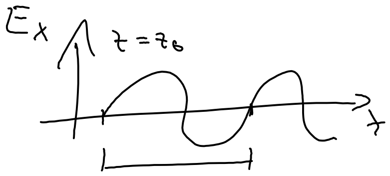
$E_x(z,t) = \hat{A}_1 \cos(\omega t - kz + \varphi_{x1})$



$k\,\Delta z = 2\pi = k\lambda$

$\Delta z = \lambda$

$k = \frac{2\pi}{\lambda}$ $\quad \lambda$ Wellenlänge



$\omega\Delta t = 2\pi = \omega T$

$\omega = \frac{2\pi}{T}$

T: Periodendauer

$f = \frac{1}{T}$

$k = \omega\sqrt{\varepsilon\mu} = \frac{\omega}{c} = \frac{2\pi}{\lambda} = \frac{2\pi f}{c}$

$\boxed{c = \lambda f}$

$c = \frac{\omega}{k}$ Phasengeschwindigkeit

$\vec{k} = k\vec{n}$ $\qquad$ Wellenzahlvektor

10.4 Phasengeschwindigkeit

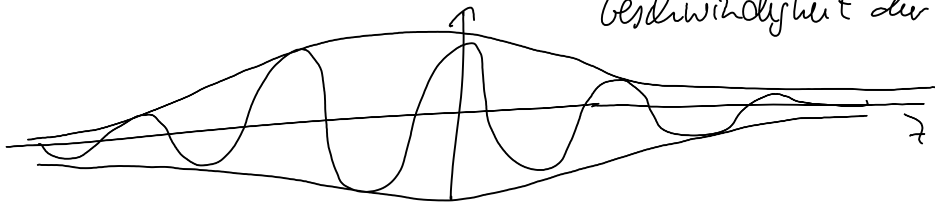
$v_{ph} = c = \frac{1}{\sqrt{\varepsilon\mu}} \quad : \sigma \ll \omega\varepsilon \qquad \varepsilon, \mu = \text{const}$

$\varepsilon(\omega), \mu(\omega) \qquad \sigma \not\ll \omega\varepsilon$

$v_{ph} = v_{ph}(\omega) = v_{ph}(k)$

Gruppengeschwindigkeit: v_g

Geschwindigkeit der Information



$v_g = v_{ph} = c$

$$f(z,t) = \mathrm{Re}\{\int \underline{A}(k')\, e^{j(\omega'(k')t - k'z)}\, dk'\}$$

$k \gg \Delta k$

$$f(z,t) = \mathrm{Re}\{\int_{k-\Delta k}^{k+\Delta k} \underline{A}(k')\, e^{j(\omega' t - k'z)}\, dk'\} \qquad k' = k + \delta k$$

Taylor

$$= \mathrm{Re}\{\int_{-\Delta k}^{\Delta k} \underline{A}(k+\delta k)\, e^{j((\omega+\delta\omega)t - (k+\delta k)z}\, d\,\delta k\} \qquad \omega(k+\delta k) \approx \omega(k) + \underbrace{\frac{d\omega}{dk}\delta k}_{\delta\omega}$$

$$f(z,t) = \mathrm{Re}\{e^{j(\omega t - kz)} \int_{-\Delta k}^{\Delta k} \underline{A}(k+\delta k)\, e^{j\overbrace{(\delta\omega t - \delta k z)}^{\varphi_E}}\, d\,\delta k\}$$

$$\left.\frac{dz}{dt}\right|_{\text{Einhüllende}} = \frac{\delta\omega}{\delta k} = \frac{d\omega}{dk} = v_g$$

$\omega = v_{ph} \cdot k$

$$v_g = \frac{d\omega}{dk} = v_{ph} + k\,\frac{dv_{ph}}{dk} = v_{ph} + k\,\frac{dv_{ph}}{d\omega}\underbrace{\frac{d\omega}{dk}}_{v_g}$$

$$\boxed{v_g = \frac{v_{ph}}{1 - k\,\frac{dv_{ph}}{d\omega}}}$$

$\frac{d\varepsilon}{d\omega} > 0$ normale Dispersion

$\frac{dv_{ph}}{d\omega} < 0 \Rightarrow v_g < v_{ph} < c_0$

$\frac{d\varepsilon}{d\omega} < 0$

11 Leistungsbilanzen im EM-Feld

Energiesatz: $\mathrm{rot}\,\vec{H} = \vec{J} + \dot{\vec{D}} \quad |\cdot\vec{E} \qquad \vec{E}\,\mathrm{rot}\,\vec{H} = \vec{E}\vec{J} + \vec{E}\dot{\vec{D}}$

$\mathrm{rot}\,\vec{E} = -\dot{\vec{B}} \quad |\cdot\vec{H} \qquad \vec{H}\,\mathrm{rot}\,\vec{E} = -\vec{H}\dot{\vec{B}}$

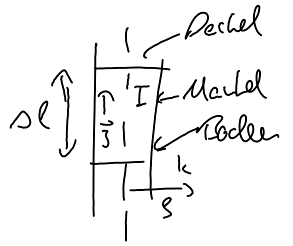
$$\vec{E}\,\mathrm{rot}\vec{H} - \vec{H}\,\mathrm{rot}\vec{E} = \vec{E}\vec{J} + \vec{E}\dot{\vec{D}} + \vec{H}\dot{\vec{B}} = -\mathrm{div}\,(\vec{E}\times\vec{H})$$

$$\mathrm{div}(\vec{E}\times\vec{H}) = \nabla(\vec{E}\times\vec{H}) = \nabla(\vec{E}\times\vec{H}) + \nabla(\vec{E}\times\vec{H})$$

$$= \vec{H}(\nabla\times\vec{E}) - E(\nabla\times\vec{H})$$

$$-\oint_{F_V}(\vec{E}\times\vec{H})\,d\vec{F} = \int_V(\vec{E}\vec{J} + \vec{E}\dot{\vec{D}} + \vec{H}\dot{\vec{B}})\,dV$$

Beispiel: stationäres Strömungsfeld



$\vec{J} = \sigma\vec{E}$ $\quad \vec{E} = E_0\,\vec{e}_z = \frac{J}{\sigma\,F_Q}\vec{e}_z$ im Leiter

$\vec{H} = \frac{I}{2\pi\varrho}\vec{e}$ für $\varrho > \varrho_0$

$$\oint_{\text{Dose}}(\vec{E}\times\vec{H})\,d\vec{F} = \underbrace{\cancel{\int_{\substack{\text{Deckel}+\\ \text{Boden}}}(\vec{E}\times\vec{H})\,d\vec{F}}}_{\mathrm{I}} + \underbrace{\int_{\text{Mantel}}(\vec{E}\times\vec{H})\,d\vec{F}}_{\mathrm{II}}$$

$$\mathrm{I})\int_{\substack{\text{Deckel}\\ \text{Boden}}}(E_z\vec{e}_z \times H_\varphi\vec{e}_\varphi)\,dF(\pm\vec{e}_z) = 0$$

$$\mathrm{II})\int_{\text{Mantel}}(\vec{E}\times\vec{H})\,d\vec{F} = \int_{\text{Mantel}}((E_z\vec{e}_z)\times(H_\varphi\vec{e}_\varphi))\,dF\,\vec{e}_\varrho$$

$$+\int_{\text{Mantel}}\underbrace{((E_\varphi\vec{e}_\varphi \times H_\varphi\vec{e}_\varphi)}_{=0}\,dF\,\vec{e}_\varrho$$

$$+\int_{\text{Mantel}}\underbrace{((E_\varrho\vec{e}_\varrho)\times H_\varphi\vec{e}_\varphi)}_{=0}\,dF\,\vec{e}_\varrho$$

$$\mathrm{Rot}\,\vec{E} = 0 \Rightarrow \vec{E}(\varrho_0)\ \text{stetig} = E_z(\varrho_0)\,\vec{e}_z = \frac{I}{\sigma\,F_Q}\vec{e}_z$$

$$-\int_0^{\Delta l}\int_0^{2\pi}\left(\frac{I}{\sigma F_Q}\cancel{\vec{e}_z}\times\frac{I}{2\pi\cancel{\varrho_0}}\cancel{\vec{e}_\varphi}\right)\cancel{\varrho_0}\,d\varphi\,dz\,\cancel{\vec{e}_\varrho} = -\frac{\Delta l}{\sigma F_Q}I^2 = -I^2R$$

$$-\oint_F(\vec{E}\times\vec{H})\,d\vec{F} = P_{\text{Joule}}$$

$$\vec{J} = \sigma(\vec{E} + \vec{E}^{(e)})$$

$$\vec{E}\vec{J} = \underbrace{\frac{\vec{J}^2}{\sigma}}_{p_{Joule}} - \underbrace{\vec{J}\vec{E}^{(e)}}_{\text{Quellenleistungsdichte } p^{(e)}}$$

$$\frac{\partial \varepsilon}{\partial t} = 0 \ ; \ \frac{\partial \mu}{\partial t} = 0$$

$$\vec{H}\dot{\vec{B}} = \vec{H}\frac{\partial}{\partial t}(\mu\vec{H}) = \mu\vec{H}\frac{\partial}{\partial t}\vec{H} = \frac{1}{2}\mu\frac{\partial \vec{H}^2}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu\vec{H}^2\right) = \frac{\partial}{\partial t} W_{magn}$$

$$\vec{E}\dot{\vec{D}} = \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon\vec{E}^2\right) = \frac{\partial W_{el}}{\partial t}$$

$\vec{S} = \vec{E}\times\vec{H}$ Poynting-Vektor

$$p^{(e)} = p_{Joule} + \frac{d}{dt}(W_{magn} + W_{el}) + \operatorname{div}\vec{S}$$

Für $p^{(e)} = p_{Joule} = 0$

$$\oint_{F_V} \vec{S}\, d\vec{F} = -\frac{d}{dt}(W_{el} + W_{magn}) \qquad W_{el} = \int_V \frac{1}{2}\varepsilon\vec{E}^2 dV$$

Ebene, linear polarisierte Welle

$$c = \frac{1}{\sqrt{\varepsilon\mu}} \qquad \vec{E} = E_x(ct - z)\vec{e}_x$$

$$\vec{H} = H_y(ct - z)\vec{e}_y = \frac{E_x}{Z}\vec{e}_y$$

$$\vec{S} = \vec{E}_x \times \vec{H} = E_x H_y \vec{e}_z = \frac{E_x^2}{Z}\vec{e}_z = \vec{S}(ct - z)$$

$$W_{EM} = \frac{\varepsilon}{2}E_x^2 + \frac{\mu}{2}H_x^2 = \varepsilon E_x^2$$

$$W_L v_{Ph} = W_{EM} c = S \qquad ?!?$$

<u>HEZ:</u> $\vec{E} = \mathrm{Re}\{\underline{\vec{E}}e^{j\omega t}\}$ $\qquad \vec{H} = \mathrm{Re}\{\underline{\vec{H}}e^{j\omega t}\}$

$$\vec{S} = \vec{E}\times\vec{H} = \frac{1}{2}\{\underline{\vec{E}}e^{j\omega t} + \underline{\vec{E}}^* e^{-j\omega t}\}\times\frac{1}{2}\{\underline{\vec{H}}e^{j\omega t} + \underline{\vec{H}}^* e^{-j\omega t}\}$$

$$= \frac{1}{4}\left[\underline{\vec{E}}\times\underline{\vec{H}}e^{j2\omega t} + \underline{\vec{E}}^*\times\underline{\vec{H}}^* e^{-2j\omega t} + \underline{\vec{E}}\times\underline{\vec{H}}^* + \underline{\vec{E}}^*\times\underline{\vec{H}}\right]$$

$$\overline{\vec{S}} = \frac{1}{T}\int_0^T \vec{S}(t)\,dt = \frac{1}{2}\mathrm{Re}\{\underline{\vec{E}}\times\underline{\vec{H}}^*\}$$

$$\vec{S} = \frac{1}{2}\, Re\{\underline{\vec{S}}\} \qquad \underline{\vec{S}} = \underline{\vec{E}} \times \underline{\vec{H}}^*$$

$$-\frac{1}{2}\, div\,(\underline{\vec{S}}) = -\frac{1}{2}\, div\,(\underline{\vec{E}} \times \underline{\vec{H}}^*)$$

$$= -\frac{1}{2} \underbrace{rot\, \underline{\vec{E}}}_{-j\omega\underline{\vec{D}}}\, \underline{\vec{H}}^* + \frac{1}{2}\, \underline{\vec{E}}\, \underbrace{rot(\underline{\vec{H}}^*)}_{=(\underline{\vec{J}}+j\omega\underline{\vec{D}})^* = (\sigma + j\omega\underline{\varepsilon})^* \underline{\vec{E}}^*}$$

$$= j\omega\underline{\mu}\, \frac{\underline{\vec{H}}\,\underline{\vec{H}}^*}{2} + (\sigma + j\omega\underline{\varepsilon})^* \frac{\underline{\vec{E}}\,\underline{\vec{E}}^*}{2}$$

$$\underline{\varepsilon} = \varepsilon' - j\varepsilon''$$
$$\underline{\mu} = \mu' - j\mu''$$

$$= \sigma\, \frac{\underline{\vec{E}}\,\underline{\vec{E}}^*}{2} + \omega\varepsilon''\, \frac{\underline{\vec{E}}\,\underline{\vec{E}}^*}{2} + \omega\mu''\, \frac{\underline{\vec{H}}\,\underline{\vec{H}}^*}{2}$$

$$+ 2j\omega \left(\underbrace{\mu' \frac{\underline{\vec{H}}\,\underline{\vec{H}}^*}{4}}_{W_{magn}} - \underbrace{\varepsilon' \frac{\underline{\vec{E}}\,\underline{\vec{E}}^*}{4}}_{W_{el}} \right)$$