Übung 7

Donnerstag, 15. Juli 2010
00:14

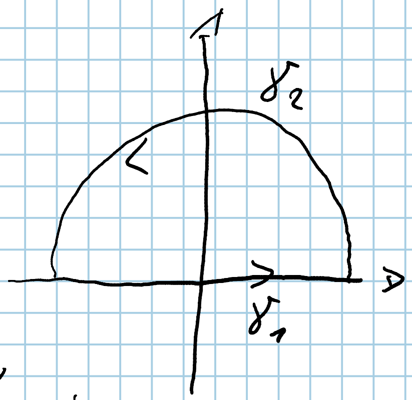
Aufg. 32

a) $\int_{-\infty}^{\infty} \frac{2}{x^2-2x+5}\,dx$

Idee:

Residuensatz anwenden

Dazu: Def. $f(z) = \frac{2}{z^2-2z+5}$ und integrieren f entlang Γ



$\gamma_1(t) = t,\ t \in [-R, R]$

$\gamma_2(t) = R \cdot e^{it},\ t \in [0, \pi]$

f hat isol. Singularitäten in

$z_{1,2} = 1 \pm \sqrt{1-5} = 1 \pm 2i$

Dabei liegt nur $z_1 = 1+2i$ in $\mathrm{int}(\Gamma)$ für $(R > \sqrt{5})$

z_1 ist Polstelle 1. Ordnung von f und

$$Res(f, z_1) = \frac{2}{2z-2}\Big|_{z=z_1}$$

$$= \frac{2}{2+4i-2} = \frac{1}{2i}$$

Da Γ einfach geschlossen, st. st. diff'bar, f holomorph in $\mathrm{int}(\Gamma)$ bis auf z_1

$$\underset{\text{Res. Satz}}{\Longrightarrow} \int_\Gamma f(z)\,dz = 2\pi i \cdot \mathrm{Res}(f, z_1)$$

$$= 2\pi i \cdot \frac{1}{2i} = \pi$$

Integral entlang γ_2:

$$\left| \int_{\gamma_2} f(z)\,dz \right| = \left| \int_0^\pi f(R\cdot e^{it})\, R\cdot i\cdot e^{it}\,dt \right|$$

$$\leq R \cdot \int_0^\pi |f(R\cdot e^{it})|\,dt$$

$$\leq \underbrace{R \cdot \max_{t\in[0,\pi]} |f(R\cdot e^{it})|}_{\leq C\cdot R^{-2}}\, \pi$$

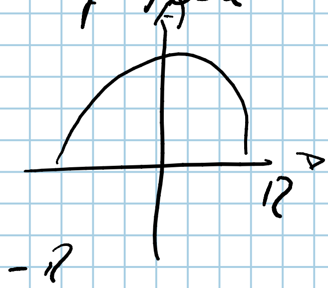
$$\leq C\cdot \pi \cdot R^{-1} \xrightarrow{R\to\infty} 0$$

Damit $\lim\limits_{R\to\infty} \int_\Gamma f(z)\,dz = \int_{-\infty}^{\infty} \frac{2}{x^2-2x+5}\,dx = \underline{\underline{\pi}}$

b) $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} \, dx$

Def $f(z) = \frac{z^2}{(z^2+1)^2}$, dann hat f Pole 2. Ordnung in $\pm i$

Integriere f entlang Γ mit



Für $R > 1$ liegt i in $\text{int}(\Gamma)$

Da Γ einfach geschl. st. st. diff'bar.
f holomorph in $\text{int}(\Gamma)$ mit Ausnahme von i

$$\overset{RS}{\Longrightarrow} \int_\Gamma f(z)\,dz = 2\pi i \cdot \text{Res}(f, i) \qquad \cdot \frac{1}{(2-1)!}$$

$$= 2\pi i \left(\frac{z^2}{(z+i)^2} \right)' \Bigg|_{z=i}$$

$$= 2\pi i \; \frac{2z(z+i) - z^2 \cdot 2}{(z+i)^3} \Bigg|_{z=i}$$

$$= 2\pi i \cdot \frac{-i}{4} = \frac{\pi}{2}$$

Entlang KB

$$\left|\int_0^{\pi} \frac{R^2 e^{2it}}{(R^2 e^{2it}+1)^2}\, R\cdot i\cdot e^{it}\,dt\right| \le R\pi\cdot R^{-2}\cdot C$$

$$= C\cdot R^{-1} \xrightarrow{R\to\infty} 0$$

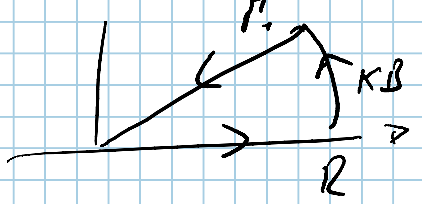
Für $R\to\infty$ erhalten wir

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx = \frac{\pi}{2}$$

Aufg. 33

$$\int_{-\infty}^{\infty} \frac{1}{1+x^{10}}\,dx$$ „Tortenstückintegral"

Γ:



Def. $f(z) = \dfrac{1}{1+z^{10}}$

f hat in int Γ die NS $z_1 = e^{i\frac{\pi}{10}}$

z_1 ist einfache PS

$$\mathrm{Res}(f, z_1) = \frac{1}{10\cdot z^9}\Big|_{z=z_1} = \frac{1}{10}\,e^{-\frac{9i\pi}{10}}$$

i) KB

$$\left| \int_{KB} f(z)\, dz \right| = \left| \int_0^{\pi/5} \frac{1}{1+R^{10} e^{10ti}} \cdot R \cdot i\, e^{it}\, dt \right|$$

Γ $\gamma(t) = R \cdot e^{it}$, $t \in [0, \pi/5]$ $\lrcorner$

$$\leq R \cdot \pi/5 \cdot C \cdot R^{-10} = C \cdot R^{-9} \xrightarrow{R \to \infty} 0$$

ii) Γ_1 : $\gamma_1(t) = \underbrace{(R-t)}_{R} \cdot e^{i\frac{\pi}{5}}$, $t \in [0, R]$

$$\int_{\Gamma_1} f(z)\, dz = \int_0^{R} \frac{1}{1+(R-t)^{10} e^{2\pi i}} \left(-e^{i\pi/5}\right) dt$$

$$= -e^{i\frac{\pi}{5}} \int_0^{R} \frac{1}{1+(R-t)^{10}}\, dt$$

$$\underset{x = R-t}{=} -e^{i\pi/5} \int_R^0 \frac{1}{1+x^{10}}\, dx = -e^{i\pi/5} \int_0^R \frac{1}{1+x^{10}}\, dx$$

Insgesamt

$$\int_\Gamma f(z)\,dz = \int_0^R \frac{1}{1+x^{10}}\,dx + \int_{KB} f(z)\,dz - e^{i\frac{\pi}{5}} \int_0^R \frac{1}{1+x^{10}}\,dx$$

Da Γ einfach geschl. st. st. diff, f hol in $\mathrm{int}(\Gamma)$ mit Ausnahme von z_1

$$\overset{RS}{\Rightarrow} \int_\Gamma f(z)\,dz = 2\pi i \;\mathrm{Res}(f, z_1)$$

$$= 2\pi i \,\frac{1}{10}\, e^{-\frac{9 i\pi}{10}} = -\frac{\pi i}{5}\, e^{\frac{i\pi}{10}}$$

$$\Rightarrow \lim_{R\to 0} \int_\Gamma f(z)\,dz = \underbrace{\left(1-e^{i\frac{\pi}{5}}\right)}_{e^{\frac{i\pi}{10}}\left(e^{-\frac{i\pi}{10}}-e^{\frac{i\pi}{10}}\right)} \int_0 \frac{1}{1+x^{10}}\,dx$$

$$= e^{i\frac{\pi}{10}}(-2i)\sin\left(\frac{\pi}{10}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{1+x^{10}}\,dx = \frac{\pi}{10}\cdot\frac{1}{\sin\left(\frac{\pi}{10}\right)}$$

Aufg. 34

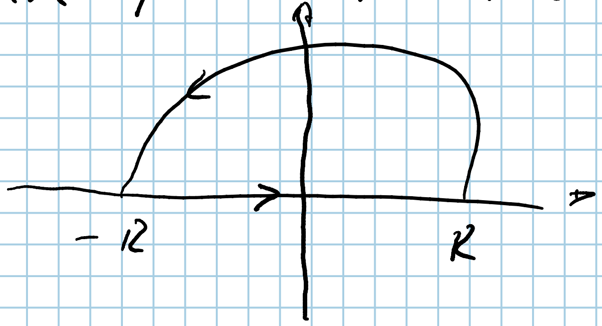
$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+x+1}\,dx \qquad \text{Def. } f(z) = \frac{e^{iz}}{z^2+z+1}$$

f hat isol. Singularitäten

in $z^2+z+1=0 \iff z_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4}-1}$

$$\Rightarrow z_{1,2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}}$$

Integriere f über Γ mit Γ:



In int(Γ) liegt $z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i = e^{i\frac{2\pi}{3}}$

z_1 ist einfache PS mit $\mathrm{Res}(f, z_1) = \frac{e^{iz_1}}{2z_1+1}$

$$= e^{-\frac{i}{2} - \frac{\sqrt{3}}{2}} \cdot \frac{1}{i\sqrt{3}}$$

Integral über KB

$$\left|\int_{KB} f(z)\,dz\right| = \left|\int_0^{\pi} \frac{e^{Ri(\cos t + i\sin t)}}{R^2 e^{2it} + R\cdot e^{it} + 1}\, i\cdot R\cdot e^{it}\, dt\right|$$

$$\leq \int_0^{\pi} \frac{e^{-R\cdot\sin t}\cdot R}{|R^2 e^{2it} + Re^{it} + 1|}\, dt < R^{-1}\pi c \xrightarrow{R\to\infty} 0$$

Γ einfach geschl. st. st diff'bar, f hol. in $\operatorname{int}(\Gamma)$ mit Ausnahme von z_1

$$\underset{RS}{\Rightarrow} \int_{\Gamma} f(z)\,dz = 2\pi i \operatorname{Res}(f, z_1) = 2\pi i \frac{1}{i\sqrt{3}} e^{-\frac{i}{2} - \frac{\sqrt{3}}{2}}$$

$$\Rightarrow \lim_{R\to\infty} \int_{\Gamma} f(z)\,dz = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + x + 1} = \frac{2\pi}{\sqrt{3}} e^{-\frac{i}{2} - \frac{\sqrt{3}}{2}}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + x + 1}\, dx = \operatorname{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + x + 1}\, dx\right)$$

$$= \operatorname{Re}\left(\frac{2\pi}{\sqrt{3}} e^{-\frac{\sqrt{3}}{2}} \left(\cos(1/2) - i\sin(1/2)\right)\right)$$

$$= \frac{2\pi}{\sqrt{3}} e^{-\frac{\sqrt{3}}{2}} \cos(1/2)$$

Aufg. 35

$$\int_0^\infty \frac{\log x}{1+x^2}\,dx$$ setze f holom. fort auf

$\mathbb{C} \setminus \{ix,\ x \in \mathbb{R},\ x < 0\}$

Def. $f(z) = \frac{\log z}{1+z^2}$, f hol. außer in

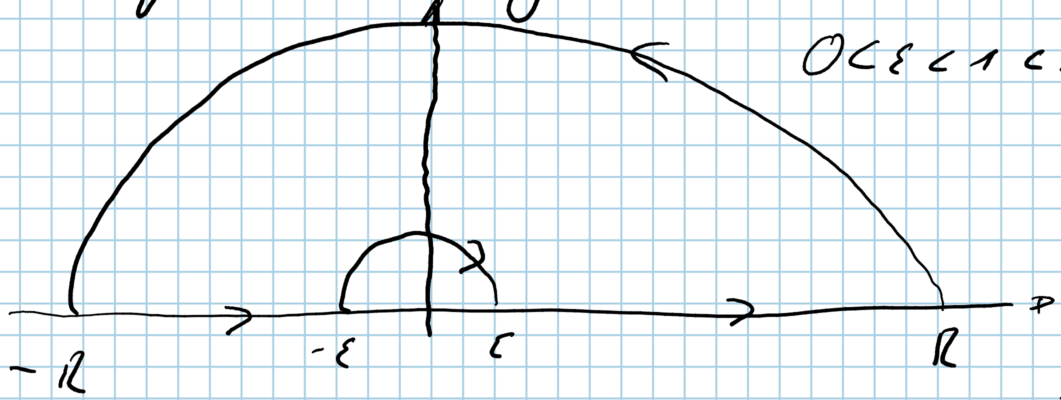
$\{z \in \mathbb{C} \mid \operatorname{Im} z < 0\}$

f hat in $\{z \in \mathbb{C} \mid \operatorname{Im} z < 0\}$ iso. Sing. in i

In i hat f einen einfachen Pol und

$$\operatorname{Res}(f, i) = \frac{\log i}{2i} = \frac{\pi}{4}$$

Integriere f entlang Γ mit



$0 < \varepsilon < 1 < R$

Da Γ einfach geschlossen, d.h. diffbar
f bis auf $z = i$ in $\operatorname{int}(\Gamma)$ holom.

$$\overset{\text{Res}}{\Longrightarrow} \int_\Gamma f(z)\,dz = 2\pi i \operatorname{Res}(f, i) = 2\pi i \frac{\pi}{4} = \frac{i\pi^2}{2}$$

i) entlang Γ_R

$$\left| \int_{\Gamma_R} f(z)\,dz \right| = \left| \int_0^\pi iRe^{it} \frac{\log(R \cdot e^{it})}{1 + R^2 e^{2it}}\,dt \right|$$

$$\leq R \cdot \int_0^\pi \frac{|\log R + it|}{|1 + R^2 e^{2it}|} \leq R\pi \frac{\log R + \pi}{?} \xrightarrow{R \to \infty} 0$$

ii) Entlang Γ_ε

$$\left| \int_{\Gamma_\varepsilon} f(z)\, dz \right| \leq \varepsilon \int_0^{\pi} \frac{|\log \varepsilon + it|}{|1 + \varepsilon^2 \cdot e^{2it}|}\, dt$$

$$\leq \pi \cdot \varepsilon \cdot \frac{\log \varepsilon + \pi}{1 - \varepsilon^2} \xrightarrow{\varepsilon \to 0} 0$$

iii) $$\int_{-R}^{-\varepsilon} f(z)\, dz = \int_{\varepsilon}^{R} f(-t)\, dt$$

$$= \int_{\varepsilon}^{R} \frac{\log(-t)}{1+t^2}\, dt = \int_{\varepsilon}^{R} \frac{\log(-1) + \log(t)}{1+t^2}\, dt$$

$$= \int_{\varepsilon}^{R} f(t)\, dt + \int_{\varepsilon}^{R} \frac{i\pi}{1+t^2}\, dt$$

$$= \int_{\varepsilon}^{R} f(t)\, dt + i\pi \left[\arctan(t)\right]_{\varepsilon}^{R}$$

$$\Rightarrow \int_{\Gamma} f(z)\, dz = \int_{-R}^{-\varepsilon} f(t)\, dt + \int_{\Gamma_\varepsilon} f(z)\, dz + \int_{\varepsilon}^{R} f(t)\, dt + \int_{\Gamma_R} f(z)\, dz$$

$$= 2 \cdot \int_{\varepsilon}^{R} f(t)\, dt + i\pi \left(\arctan(R) - \arctan(\varepsilon)\right)$$

$$+ \int_{\Gamma_\varepsilon} f(z)\, dz + \int_{\Gamma_R} f(z)\, dz$$

Für $R \to \infty$ und $\varepsilon \to 0$ erhalten wir

$$2 \int_0^\infty \frac{\log(x)}{x^2+1}\, dx + i\pi \frac{\pi}{2} = \frac{i\pi^2}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\log x}{x^2+1}\, dx = 0$$