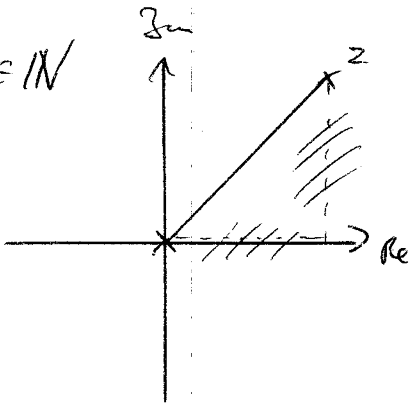
30. JUNI 2010

HM4 KGÜ 5

B23.)

$$\int_0^z \zeta^n \, d\zeta \qquad \xleftarrow{} f(\zeta) \qquad z \in \mathbb{C} \qquad n \in \mathbb{N}$$



$$\Gamma = \overline{0z} = \{ \zeta \in \mathbb{C} \mid \zeta = t \cdot z \; ; \; t \in [0,1] \}$$

$$= \int_0^1 t^n z^n \, z \, dt$$

$$= z^{n+1} \int_0^1 t^n \, dt \quad = \frac{z^{n+1}}{n+1} := F(z)$$

Stammfunktion?

$$\frac{d}{dz} F(z) = \left(z^{n+1} \frac{1}{n+1} \right)' = z^n \frac{n+1}{n+1} = z^n = f(z) \checkmark$$

B24.)

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it}) \, dt$$

mit Cauchy-Integralformel:

$\gamma : [0, 2\pi] \to \mathbb{C} \qquad \gamma(t) = z_0 + Re^{it} \;\hat{=}\;$ Parameter-Darstellung vom $B_r(z_0)$

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(z)}{z - z_0} \, dz$$

$$= \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{\cancel{z_0 + Re^{it} - z_0}} \cancel{\cdot i Re^{it}} \, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it}) \, dt$$

A25.1) $f: \mathbb{C} \to \mathbb{C}$ holomorph

$$f(\cancel{x+} iy) = \tilde{f}(x) + i\tilde{f}(y) \qquad f = \tilde{f}$$

zu zeigen: $f(z) = \tilde{f}(1) \cdot z$

(1) zeige f linear:

a.) $z_0 = x_0 + iy_0 \qquad h \in \mathbb{R} \qquad z = z_0 + h$

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{f(z_0 + h) - f(z_0)}{z_0 + h - z_0}$$

$$= \frac{\tilde{f}(x_0 + h) + \cancel{i\tilde{f}(y_0)} - \tilde{f}(x_0) - \cancel{i\tilde{f}(y_0)}}{h}$$

$$= \frac{\tilde{f}(x_0 + h) - \tilde{f}(x_0)}{h} \xrightarrow{h \to 0} \tilde{f}'(x_0)$$

b.) ebenso für $z = z_0 + ih$

$$\frac{f(z) - f(z_0)}{ih} = \frac{\tilde{f}(y_0 + h) - \tilde{f}(y_0)}{h} \xrightarrow{h \to 0} \tilde{f}(y_0)$$

c.) da f holomorph, gilt Cauchy-Riemann:

$$\tilde{f}'(x_0) = \tilde{f}'(y_0) \qquad \forall x_0, y_0 \in \mathbb{R}$$

variiere $x_0 \Rightarrow x$

$$\Rightarrow \tilde{f}'(x) = \tilde{f}'(y_0) = \text{const}$$

ebenso:

$$\tilde{f}'(y) = \text{const}, \qquad \forall y \in \mathbb{R}$$

$$\Rightarrow \tilde{f}(x) = a \cdot x + b \qquad \forall x \in \mathbb{R}, \quad a, b \in \mathbb{C}$$

f holomorph $\to$ Identitätssatz Th 4.7. liefert:

$$\Rightarrow f(z) = az + b \qquad z \in \mathbb{C}$$

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② zeige $b=0$:

für $x \in \mathbb{R}$, $y=0$ gilt: $\cancel{f(x)}$

$f(x) = \tilde{f}(x) + i\tilde{f}(0)$ wobei $f \equiv \tilde{f}$

$\Rightarrow 0 = \tilde{f}(0) = a \cdot 0 + b$

$\Rightarrow b = 0$

$\Rightarrow f(z) = az$

③ zeige $a = f(1)$:

Sei $z \in \mathbb{C}$

$\tilde{f}(1) \cdot z = a \cdot 1 \cdot z = az = f(z)$ ✓

$\Rightarrow \underline{\underline{f(z) = f(1) \cdot z}}$

<u>B 26.)</u>

für $|z|>1$ ist der Integrand holomorph

$\Rightarrow$ Cauchy-Integralsatz liefert:

$$\int_{|\zeta|=1} \frac{\sin(2\zeta)}{(\zeta-z)^2}\, d\zeta = 0$$

$F(z) \equiv 0$

$F'(z) = 0$

<u>für $|z|<1$:</u>

Cauchy Integralformel

$$f^{(n)} = \underbrace{\frac{n!}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{(\zeta-z)^{n+1}}\, d\zeta}_{}$$

$F(z)$ für $n=1$, Γ ist Einheitskreis, $f(\zeta) = \sin(2\zeta)$

$$F(z) = 2\pi i \cdot f'(z)$$
$$= 2\pi i \cdot 2\cos(2\cancel{z})$$
$$= -8\pi i \sin^2(\cancel{z})$$

$$\Rightarrow F'(z) = \begin{cases} -8\pi i \sin(2z) & |z|<1 \\ 0 & |z|>1 \end{cases}$$

$$\Rightarrow F'\left(\frac{\pi}{4}\right) = -8\pi i \sin\left(\frac{\pi}{2}\right) = -8\pi i$$
$$F'\left(\frac{\pi}{2}\right) = 0$$

<u>A 27.)</u>

$$\oint f(z) \xrightarrow{\text{Parametrisieren}} \int \dots dt + i\int \dots dt$$

$\oint f(z) \rightarrow$ Cauchy Integralformel $\rightarrow$ Lösung

$$e^{it} = \cos(t) + i\sin(t)$$

$|z|=1 \to \gamma(t) = e^{it}$ $t \in [0, 2\pi]$

$f(z) = e^z / z^{n+1}$ holomorph für $z \neq 0$

$$\int_\Gamma f(\gamma(t))\,\gamma'(t)\,dt = \int_0^{2\pi} e^{e^{it}} \cdot (e^{it})^{-(n+1)} \cdot i \cdot e^{it}\,dt$$
$$= i\int_0^{2\pi} e^{e^{it} + \sin(t)} \cdot e^{-int}\,dt$$
$$= i\int_0^{2\pi} e^{\cos(t)} \cdot e^{i(\sin(t) - nt)}\,dt$$
$$= i\int_0^{2\pi} e^{\cos(t)}\left(\cos(\sin(t) - nt) + i\sin(\sin(t) - nt)\right)dt$$
$$= i\int_0^{2\pi} e^{\cos(t)}\cos(nt - \sin(t))\,dt + \int_0^{2\pi} e^{\cos(t)}\sin(nt - \sin(t))\,dt$$

$$\int_{|z|=1} \frac{e^z}{z^{n+1}}\, dz = \int_{|z|=1} \frac{g(z)}{(z-0)^{n+1}}\, dz \qquad g(z) = e^z$$

$$\text{CIF} \to \;= \frac{2\pi i}{n!}\left[\left(\frac{d}{dz}\right)^n e^z\right]_{z=0}$$

$$= \frac{2\pi}{n!}$$

Vergleich von Real- und Imaginärteil

$$\text{Real: } \int_0^{2\pi} e^{\cos(t)} \sin(nt - \sin(t))\, dt = 0$$

$$\text{Imaginär: } \int_0^{2\pi} e^{\cos(t)} \cos(nt - \sin(t))\, dt = \frac{2\pi}{n}$$

<u>II 28.1</u>

$f(z) = \frac{e^{3z}}{(z-2)^2}$ ist holomorph für $0<|z-2|<\infty$

$\boxed{e^x = \sum \frac{x^n}{n!}}$

Laurent-Reihe $z=2$:

$$f(z) = \frac{1}{(z-2)^2}\, e^{3(z-2+2)} = \frac{e^6}{(z-2)^2} \sum_{n=0}^{\infty} \frac{3^n}{n!} (z-2)^n$$

$$= \sum_{n=0}^{\infty} \frac{e^6\, 3^n}{n!} (z-2)^{n-2}$$

für $|z-2| > 0$