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07. JUNI 2010

A3.17.)

$$A = \begin{pmatrix} \cancel{2} & -1 & 0 & 0 \\ -1 & 2 & -\alpha & 0 \\ 0 & -\alpha & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & \alpha & 0 \\ 0 & \alpha & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

cholesky $\Rightarrow L D L^T$, $D$ Diag, $D_{ii} > 0$

$$d_{k,k} = a_{k,k} - \sum_{j=1}^{k-1} l_{k,j}^2 \cdot d_{j,j}$$

$$l_{i,k} = \left(a_{i,k} - \sum_{j=1}^{k-1} l_{i,j} \cdot d_{j,j} \cdot l_{k,j}\right) / d_{k,k}$$

**k=1:**

$$d_{k,k} = d_{1,1} = a_{1,1} - \sum_{j=1}^{0} \dots = \underline{4}$$

leere Summen

**i=2:**

$$l_{2,1} = \left(a_{2,1} - \sum_{j=1}^{0} \dots\right) / d_{1,1} = \underline{\frac{1}{4}}$$

**i=3, i=4** $\Rightarrow l_{3,1} = l_{4,1} = 0$

**k=2:**

$$d_{2,2} = a_{2,2} - \sum_{j=1}^{1} l_{2,j}^2 \cdot d_{j,j} = 4 - \left(\frac{1}{4}\right)^2 \cdot 4 = \underline{\frac{15}{4}}$$

$\underline{i=3:}$ $l_{3,2} = \left(a_{3,2} - \sum_{j=1}^{1} l_{3,j} \cdot d_{j,j}\, l_{2,j}\right) / d_{2,2}$

$$= (\alpha - 0)/\tfrac{15}{4} = \underline{\frac{4\alpha}{15}}$$

$\underline{i=4:}$ $l_{4,2} = \underline{0}$

$\underline{k=3:}$ $d_{3,3} = a_{3,3} - \sum_{j=1}^{2} l_{3,j}^{2}\, d_{j,j}$

$$= 4 - 0^2 \cdot 4 - \frac{4^2\alpha^2}{15^2} \cdot \frac{15}{4} = \underline{4 - \frac{4\alpha^2}{15}}$$

$\underline{i=4:}$

$l_{4,3} = \left(a_{4,3} - \sum_{j=1}^{2} l_{4,j}\, d_{j,j}\, l_{3,j}\right) / d_{3,3}$

$$= \underline{\frac{1}{4 - \frac{4\alpha^2}{15}}}$$

$\underline{k=4:}$

$d_{4,4} = a_{4,4} - \sum_{j=1}^{3} d_{j,j} \cdot l_{4,j}^{2} = 4 - \frac{1}{4 - \frac{4\alpha^2}{15}}$

A positiv definit $\Leftrightarrow d_{k,k} > 0 \quad \forall k \in \{1, \dots, 4\}$

$d_{1,1} = 4 > 0 \qquad d_{2,2} = \frac{15}{4} > 0$

$d_{3,3} = 4 - \frac{4\alpha^2}{15} \overset{!}{>} 0 \Leftrightarrow 4 > \frac{4\alpha^2}{15}$

$\Leftrightarrow 15 > \alpha^2 \Leftrightarrow \underline{\sqrt{15} > |\alpha|}$

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$$d_{4,4} = 4 - \frac{1}{4 - \frac{4\alpha^2}{15}} > 0$$

$$\Leftrightarrow 4 > \frac{1}{\underbrace{4 - \frac{4\alpha^2}{15}}_{>0 \quad d_{3,3} > 0}}$$

$$\Leftrightarrow 4\left(4 - \frac{4\alpha^2}{15}\right) > 1$$

$$\Leftrightarrow 15 > \frac{16\alpha^2}{15} \Rightarrow 14{,}0625 > \alpha^2$$

$$\Leftrightarrow |\alpha| < \sqrt{14{,}0625}$$

dann a symmetrisch positiv definit.

4.6.)

| $t_i$ | $-1$ | $1$ | $2$ |
|---|---|---|---|
| $y_i$ | $1$ | $3$ | $3$ |

$$y(t) = at + b$$

$$\sum_{i=1}^{3} (y(t_i) - y_i)^2 \Rightarrow \|Ax - f\|_2 = \min$$

wichtig

$$A = \begin{pmatrix} t_1 & 1 \\ t_2 & 1 \\ t_3 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$b = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$$

Das Ausgleichsproblem lautet:

Finde $x = \cancel{\phantom{ab}} \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$

$\|Ax-b\|_2 = \min$

Givens-Rotationen, eliminiere $a_{2,1}$

$r = \sqrt{(-1)^2+(1)^2} = \sqrt{2}$

$c = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$ | $s = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\rightsquigarrow G_{1,2} = \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ [niemals speichern!]

$G_{1,2}\cdot A = \underbrace{\begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \\ 2 & 1 \end{pmatrix}}_{A_1}$; $G_{1,2}\cdot b = \underbrace{\begin{pmatrix} \sqrt{2} \\ -2\sqrt{2} \\ 3 \end{pmatrix}}_{b_1}$

nächster Schritt: eliminieren

Eliminiere $a_{3,1}$

$r = \sqrt{(\sqrt{2})^2 + 2^2} = \sqrt{6}$

$c = \frac{\sqrt{2}}{\sqrt{6}} = \frac{\sqrt{3}}{3}$ $\quad s = \frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3}$

$G_{3,1} = \begin{pmatrix} \sqrt{3}/3 & 0 & \sqrt{6}/3 \\ 0 & 1 & 0 \\ -\sqrt{6}/3 & 0 & \sqrt{3}/3 \end{pmatrix}$

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$$G_{3,1} A_1 = \underbrace{\begin{pmatrix} \sqrt{6} & \sqrt{6}/3 \\ 0 & -\sqrt{2} \\ 0 & \sqrt{3}/3 \end{pmatrix}}_{A_2} ; \quad G_{3,1}\, b_1 = \underbrace{\begin{pmatrix} \frac{4}{3}\sqrt{6} \\ -2\sqrt{2} \\ \frac{1}{3}\sqrt{3} \end{pmatrix}}_{b_2}$$

Eliminiere $a_{3,2}$ aus $A_2$

$$r = \sqrt{(-\sqrt{2})^2 + \left(\tfrac{1}{3}\sqrt{3}\right)^2} = \sqrt{\tfrac{7}{3}}$$

$$c = -\frac{\sqrt{2}}{\sqrt{\frac{7}{3}}} = -\frac{\sqrt{42}}{7}$$

$$s = \frac{1}{3}\sqrt{3} \cdot \frac{\sqrt{3}}{7} = \sqrt{7} \cdot \frac{1}{7}$$

$$G_{3,2} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -\sqrt{42}/7 & \sqrt{7}/7 \\ 0 & -\sqrt{7}/7 & -\sqrt{24}/7 \end{pmatrix}$$

$$G_{3,2} \cdot A_2 = \begin{pmatrix} \boxed{\begin{matrix} \sqrt{6} & \sqrt{6}/3 \\ 0 & \sqrt{21}/3 \end{matrix}} \\ \begin{matrix} 0 & 0 \end{matrix} \end{pmatrix} = R ; \quad G_{32} \cdot b_2 = \begin{pmatrix} \sqrt{6} \cdot \frac{4}{3} \\ \frac{13}{21}\sqrt{21} \\ \frac{1}{7} \cdot \sqrt{14} \end{pmatrix} =: \tilde{b}$$

$\tilde{R}$

Das ist der Fehler

$$\| \underbrace{G_{3,2} \cdot G_{3,1} \cdot G_{2,1}}_{Q^T} \cdot A \cdot x - \underbrace{G_{3,2}\, G_{3,1}\, G_{2,1}}_{Q^T} b \|_2$$

$$\begin{aligned} A &= QR \quad \Big| \cdot Q^T \\ Q^T A &= \underbrace{Q^T Q}_{I} R \end{aligned}$$

$$\hat{R} \cdot x = \tilde{b} \Rightarrow \left(\begin{array}{cc|c} \sqrt{6} & \sqrt{6}/3 & \frac{4}{3}\sqrt{6} \\ 0 & \sqrt{21}/3 & \frac{13}{21}\sqrt{21} \end{array}\right)$$

$$\Rightarrow x = \begin{pmatrix} 5/7 \\ 13/7 \end{pmatrix} \Rightarrow \underline{\underline{a = \frac{5}{7}}}, \quad \underline{\underline{b = \frac{13}{7}}}$$

Residuum: $\| (A \binom{a}{b} - b) \|_2 = \underline{\underline{\frac{1}{7}\sqrt{14}}}$

<u>A 4.3.)</u> Sei $v \neq 0$: $Q_v := I - 2\frac{vv^T}{v^Tv}$

<u>a.)</u> $Q_v = Q_v^T$

Es gilt: $Q_v^T = I - 2\frac{(vv^T)^T}{v^Tv}$

$$= I - 2\frac{(v^T)^T v^T}{v^T v} = I - 2\frac{vv^T}{v^Tv} \underline{\underline{= Q_v}}$$

<u>b.)</u> $Q_v$ ist regulär + $Q_v^{-1} = Q_v$

$$Q_v \cdot Q_v = \left(I - 2\frac{vv^T}{v^Tv}\right)^2$$

$$= I^{(2)} - 4 \cdot \frac{vv^T}{v^Tv} + 4\frac{v(v^Tv)v^T}{(v^Tv)^2}$$

$$= I - 4\frac{vv^T}{v^Tv} + 4\frac{vv^T}{v^Tv} \underline{\underline{= I}}$$

<u>c.)</u> $Q_{\alpha v} = Q_v \qquad \alpha \in \mathbb{R} \setminus \{0\}$

Es gilt: $Q_{\alpha v} = I - 2\frac{(\alpha v)(\alpha v^T)}{(\alpha v)^T(\alpha v)}$

$$= I - 2\frac{\cancel{\alpha^2} vv^T}{\cancel{\alpha^2} v^Tv} = I - 2\frac{vv^T}{v^Tv} \underline{\underline{= Q_v}}$$

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**d)** $Q_v y = y \iff y^T \cdot v = 0$

$$Q_v y = y \iff y - 2\frac{vv^T}{v^Tv} y = y$$

$$\iff \cancel{2\frac{vv^T}{v^Tv}}\, y = 0$$

$$\iff v\underbrace{v^T y}_{\in \mathbb{R}} = 0 \iff v^T y = 0$$

**e)** $Q_v \cdot v = -v$

$$Q_v v = v - 2\frac{vv^T}{v^Tv} \cdot v = v - 2v = -v$$

**4.6.1 Householder**

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} \qquad x = \begin{pmatrix} b \\ a \end{pmatrix}$$

$$\|Ax - b\|_2 = \min$$

$$y = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \alpha = \mathrm{sign}(y(1)) \cdot \|y\|_2 = \sqrt{3} = 1{,}7320508\ldots$$

$$v = y + \alpha \cdot e_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \sqrt{3} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2{,}73205080 8 \\ 1 \\ 1 \end{pmatrix}$$

$$Q_v = I - 2\frac{vv^T}{v^Tv} \; ; \; \beta := \frac{2}{v^Tv} = 0{,}21132486\ldots$$

$$Q_v y = -\alpha e_1 = \begin{pmatrix} -\sqrt{3} \\ 0 \\ 0 \end{pmatrix}$$

$$Q_v \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \left( I - 2\frac{vv^T}{v^Tv} \right) \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \beta vv^T \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1,1547\ldots \\ 0,943376\ldots \\ 1,943376\ldots \end{pmatrix}$$