AM4 KGÜ 4

B18.)

$f: G \to \mathbb{C}$ holomorph

G: Gebiet

$a \in G$

$R > 0$ mit $B_R(a) \subseteq G$

$0 < r < R$

$$f(a) = \frac{1}{2\pi i} \int_{|z-a|=r} \frac{f(z)}{z-a} \, dz$$

unter gleichen Voraussetzungen und $k \in \mathbb{N}$

$$f^{(k)}(a) = \frac{\partial^k f}{\partial a^k} = \frac{k!}{2\pi i} \int_{|z-a|=r} \frac{f(z)}{(z-a)^{k+1}} \, dz$$

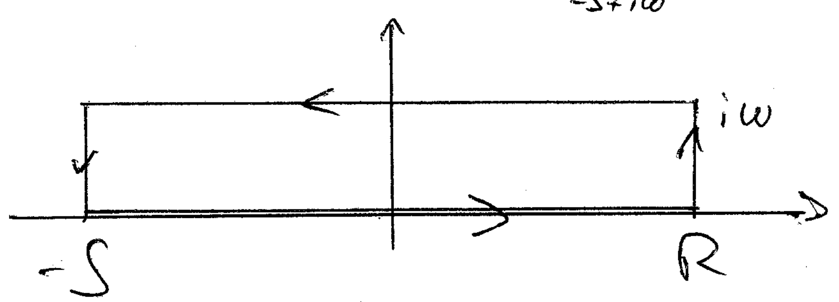
schöner wäre Theorem 4.7.

B19.)

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-ix\omega} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+i\omega)^2} \cdot e^{-\frac{\omega^2}{2}} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}} \lim_{\substack{R \to \infty \\ S \to \infty}} \int_{-S+i\omega}^{R+i\omega} e^{-\frac{z^2}{2}} \, dz$$



nach Cauchyschem Integralsatz: $\boxed{\oint f\,dz = 0 \quad \text{wenn } f \text{ holomorph in } U}$

$$\int_{\Gamma} e^{-\frac{z^2}{2}}\,dz = 0$$

rechtes Stück:

$$\left|\int_{\partial R} e^{-\frac{z^2}{2}}\,dz\right| = \left| i \int_0^\infty e^{-\frac{1}{2}(R+it)^2}\,dt\right|$$

$$= \left|\int_0^\infty e^{-\frac{R^2}{2}} \underbrace{e^{-iRt}}_{\text{phase},\ \leq 1} \cdot e^{\frac{1}{2}t^2}\,dt\right|$$

$$\leq \left|\int_0^\infty e^{-\frac{R^2}{2}} \cdot e^{\frac{1}{2}t^2}\,dt\right|$$

$$\to 0 \text{ für } R \to \infty$$

linkes Stück:

$$\left|\int_{\partial L} e^{-\frac{z^2}{2}}\,dt\right| \to 0 \text{ für } S \to \infty$$

Cauchy-Integralsatz:

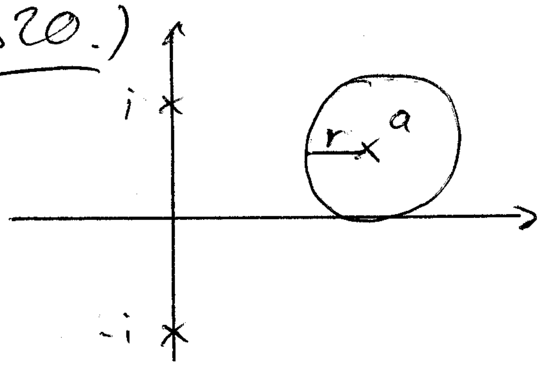
$$\Rightarrow \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx}_{=\sqrt{2\pi}} = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+i\omega)^2}\,dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+i\omega)^2}\,dt = 1$$

$$\Rightarrow \hat{f}(\omega) = e^{-\frac{\omega^2}{2}} \quad \text{q. e.d.}$$

HM4 KGÜ 4

B20.)



$|a \pm i| \neq r$ → damit γ ohne Pole

Partialbruchzerlegung

$$\frac{1}{1+z^2} = \frac{a}{z-i} + \frac{b(z-i)}{z+i} = \frac{1}{(z-i)(z+i)}$$

$z = i$ → $\underbrace{}_{0}$ $\underbrace{}_{1/2i}$

$$a = \frac{1}{2i} \qquad b = -\frac{1}{2i}$$

$$\Rightarrow \int_{|z-a|=r} \frac{1}{1+z^2}\, dz = \oint_{|z-a|=r} \left[\frac{\frac{1}{2i}}{z-i} + \frac{-\frac{1}{2i}}{z+i} \right] dz$$

mit Cauchy-Integralformel

$$= 2\pi i \cdot \begin{cases} 0 & \text{für } |a+i|>r,\ |a-i|>r \\ -\frac{1}{2i} & \text{für } |a+i|<r,\ |a-i|>r \\ +\frac{1}{2i} & \text{für } |a+i|>r,\ |a-i|<r \\ \frac{1}{2i}+\frac{-1}{2i} & \text{für } |a+i|<r,\ |a-i|<r \end{cases}$$

$$= \begin{cases} 0 \\ -\pi \\ \pi & \text{für} \quad - \text{"} - \\ 0 \end{cases}$$

B21.)

$$f(z) = (1+z)^a \qquad a \in \mathbb{C} \qquad z_0 = 0$$

$a=0$: $f(z) = 1$

$a=1$: $f(z) = 1+z$

$a \in \mathbb{N}, a \geq 2$: $\underbrace{f(z) = \sum_{k=0}^{a} \binom{a}{k} z^k}$

Binomische Formel

Summation bricht nach endlich vielen Schritten ab
↳ Konvergenzradius ∞

$a \in \mathbb{C} \setminus \mathbb{N}_0$

$$f(z) = (1+z)^a$$
$$= \exp(a \operatorname{Log}(1+z))$$

$1+z \overset{!}{>} 0 \Rightarrow z \in \mathbb{C} \setminus \{x, x \leq -1\}$

$$\Rightarrow f(z) = \sum_{k=0} a_k z^k \quad \text{für } z \in B_1(0)$$

$$\text{mit } a_k = \left[\frac{d^k}{dz^k} f(z)\right]_{z=0} \cdot \frac{1}{k!}$$

es gilt: $$\frac{d^k}{dz^k} (1+z)^a = \prod_{n=0}^{k-1} (a-n)(1+z)^{a-k}$$

$$\Rightarrow a_k = \frac{1}{k!} \prod_{n=0}^{k-1} (a-n)$$

$$\Rightarrow f(z) = \sum_{k=0}^{\infty} \frac{\prod_{n=0}^{k-1} (a-n)}{k!} z^k \quad \text{in } z \in B_1(0)$$

$$r = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty} \left| \frac{\prod_{n=0}^{k-1}(a-n)}{k!} \cdot \frac{(k+1)!}{\prod_{n=0}^{k}(a-n)} \right|$$

$$= \lim_{k\to\infty} \left|\frac{k+1}{a-k}\right| = \lim_{k\to\infty} \frac{k+1}{\sqrt{(x-k)^2+y^2}} \qquad a = x+iy$$

$$= \lim_{k\to\infty} \frac{1+\frac{1}{k}}{\sqrt{\left(\frac{x}{k}-1\right)^2 + \left(\frac{y}{k}\right)^2}} = \frac{1}{\sqrt{(-1)^2}} = \underline{1}$$

$\Rightarrow$ Konvergenzradius 1

A 22.)

$f: \mathbb{C} \to \mathbb{C}$ holomorph

$f(x+iy) = u(x,y) + i\,v(x,y)$

$u(x,y) \cdot v(x,y) \leq 0 \quad \forall (x,y) \in \mathbb{R}^2$

Satz von Liouville:

f holomorph und $|f| \leq M$

(f beschränkt) $\Rightarrow$ f konstant.

$$\left|e^{-i f^2(z)}\right| = \left|e^{-i(u^2 + 2iuv - v^2)}\right|$$

$$= \left|e^{2uv} \cdot e^{-i(u^2 - v^2)}\right|$$

$$= e^{2uv} \cdot \left|e^{-i(u^2-v^2)}\right|$$

$$= e^{2uv} \qquad u \cdot v \leq 0$$

$$\leq e^{0}$$

$$= 1 = M$$

f holomorph $\Rightarrow e^{-if^2}$ holomorph

$\Rightarrow e^{-if^2}$ konstant (nach Liouville)

$\Rightarrow f^2$ konstant

$\Rightarrow$ f konstant, wenn f holomorph

$\Rightarrow$ f konstant.