Höma 4

– Ü –

Cauchy-Integralsatz: (Th 33) $G \subset \mathbb{C}$ Gebiet

$f: G \to \mathbb{C}$ holomorph, $\gamma: [a,b] \to G$ ssd (= stückweise stetig diff'bar)

einfach geschlossen, so dass das Innere von γ in G liegt. Dann gilt: $\int_\gamma f(z)dz = 0$

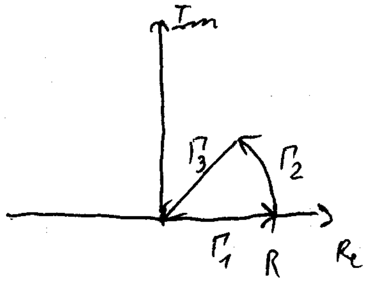
<u>A 17</u>: Berechne $\int_{-\infty}^{\infty} \cos(t^2)dt$ und $\int_{-\infty}^{\infty} \sin(t^2)dt$

Idee: Es gilt $\cos(x) = Re(e^{ix})$

$\sin(x) = Im(e^{ix})$

interpretiere die Integranden als Real- bzw. Imaginärteil ~~von~~ von e^{it^2}

Betrachte Γ



$\Gamma_1 + \Gamma_2 + \Gamma_3 = \Gamma$, $R > 0$

Γ ist stückweise stetig diff'bar, einfach geschlossen

und $int(\Gamma) \subset \mathbb{C}$,

$f(z) := e^{iz^2}$ holomorph auf $\mathbb{C}$ => CIS anwendbar

$$\int_\Gamma e^{iz^2} dz = 0 \quad, \forall R > 0$$

Berechne die Integrale über Γ_i und lasse $R \to \infty$ laufen.

<u>Γ_1</u>: $\gamma_1 [0,R] \to \mathbb{C}$

$\gamma_1(t) = t$, $\gamma_1'(t) = 1$

$$\int_{\Gamma_1} e^{-z^2} dz = \int_0^R e^{-t^2} \cdot 1\, dt \underset{R\to\infty}{\longrightarrow} \frac{\sqrt{\pi}}{2}$$

(siehe HM 3, 2)

$-\Gamma_2$: $\gamma_2 : [0, \frac{\pi}{4}] \to \mathbb{C}$, $\gamma_2(t) = Re^{it}$, $\gamma_2'(t) = iRe^{it}$

$$\int_{\Gamma_2} e^{-z^2} dz = \int_0^{\frac{\pi}{4}} e^{-(Re^{it})^2} iRe^{-it} dt \quad *$$

Behauptung: $(*) \underset{R\to\infty}{\longrightarrow} 0$

$$\left|\int_{\Gamma_2} e^{-z^2} dz\right| \leq \int_0^{\frac{\pi}{4}} \left|e^{-R^2(\cos(2t)+i\sin(2t))}\right| \cdot R \cdot \underbrace{|e^{it}|}_{=1} dt$$

$$= R \cdot \int_0^{\frac{\pi}{4}} e^{-R^2\cos(2t)} dt$$

$$= \frac{R}{2} \int_0^{\frac{\pi}{2}} e^{-R^2\cos(t)} dt$$

$$\cancel{= \frac{R}{2} \int_0 \leq \frac{R}{2} \frac{\pi}{2}}$$

$$= \frac{R}{2}\Big(\underbrace{\int_0^{\frac{\pi}{4}} e^{-R^2\cos t} dt}_{I_1} + \underbrace{\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^{-R^2\cos(t)} dt}_{I_2}\Big)$$

$$I_1 \leq \frac{R}{2} \int_0^{\frac{\pi}{4}} e^{-R^2\cos(\frac{\pi}{4})} dt = \frac{\pi}{8} R e^{-R^2 \cdot \frac{\sqrt{2}}{2}}$$

wegen $e^{R^2\frac{\sqrt{2}}{2}} > 1 + \frac{R^2\sqrt{2}}{2}$ gilt: $I_1 \leq \frac{\pi}{8} R e^{-R^2\frac{\sqrt{2}}{2}} < \frac{\pi}{8} R \frac{2}{R^2} = \frac{\pi}{4R} \underset{R\to\infty}{\longrightarrow} 0$

$$I_2 = \frac{R}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^{-R^2\cos t} dt \leq \frac{R}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{R^2 \sin t}{R^2 \sin(\frac{\pi}{4})} e^{-R^2\cos t} dt$$

$$= \frac{\sqrt{2}}{2} R^{-1} \left[e^{-R^2\cos t}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\sqrt{2}}{2} \frac{1}{R} \underset{R\to\infty}{\longrightarrow} 0$$

$$\Rightarrow \int_{\Gamma_2} e^{-z^2} dz \underset{R\to\infty}{\longrightarrow} 0$$

Hömu 4
- ä -

10.046.10

$\cdot \Gamma_3 : \gamma_3 \; [0,R] \to \mathbb{C}, \; \gamma_3(t) = e^{i\frac{\pi}{4}}(R-t)$

$\gamma_3'(t) = -e^{i\frac{\pi}{4}}$

$$\int_{\Gamma_3} e^{-z^2} dz = \int_0^R e^{-(e^{i\frac{\pi}{4}}(R-t))^2} (-e^{i\frac{\pi}{4}}) dt \quad , \; u = R-t, \; u' = -1$$

$$= e^{-i\frac{\pi}{4}} \int_R^0 e^{-(e^{i\frac{\pi}{4}}u)^2} du = \frac{1+i}{\sqrt{2}} \int_R^0 e^{-iu^2} du = \frac{1+i}{\sqrt{2}} \left[\int_R^0 \cos(u^2) - i\sin(u^2) du \right]$$

$$= \frac{1+i}{\sqrt{2}} \int_0^R -\cos(u^2) + i\sin(u^2) du$$

Insgesamt: $0 = \int_{\Gamma_1} e^{-z^2} dz + \underbrace{\int_{\Gamma_2} e^{-z^2} dz}_{\text{für } R\to\infty \; \to 0} + \int_{\Gamma_3} e^{-z^2} dz$

Das muss auch für $R \to \infty$ gelten

$$\Rightarrow \lim_{R\to\infty} \int_{\Gamma_1} e^{-z^2} dz = -\lim_{R\to\infty} \int_{\Gamma_3} e^{-z^2} dz$$

$$\Rightarrow \frac{\sqrt{\pi}}{2} = -\frac{1+i}{\sqrt{2}} \left(-\int_0^\infty \cos(t^2) dt + i \int_0^\infty \sin(t^2) dt \right)$$

$$\Leftrightarrow \frac{\sqrt{2}\pi(1-i)}{4} = \int_0^\infty \cos(t^2) dt - i \int_0^\infty \sin(t^2) dt$$

$$\Rightarrow \int_0^\infty \cos(t^2) dt = \frac{\sqrt{2}\pi}{4} \Rightarrow \int_{-\infty}^\infty \cos(t^2) dt = \frac{\sqrt{2}\pi}{2}$$

$$\Rightarrow \int_{-\infty}^\infty \sin(t^2) dt = \frac{\sqrt{2}\pi}{2}$$

Cauchy-Integral-Formel

$f: G \to \mathbb{C}$ holomorph, $a \in G$, $R > 0$, s.d.

$\overline{B_R(a)} \subseteq G$, $0 < r < R$

Dann $f(a) = \frac{1}{2\pi i} \int_{|z-a|=r} \frac{f(z)}{z-a} \, dz$

bzw. $f^{(k)}(a) = \frac{k!}{2\pi i} \int_{|z-a|=r} \frac{f(z)}{(z-a)^{k+1}} \, dz$, $\forall k \in \mathbb{N}$

A18

a) Zeige: $\int_{|z|=2} \frac{1}{z^2(z^2+5)} \, dz = 0$

[in CIF: $a = 0$, $r = 2$]

$$\frac{1}{z^2(z^2+5)} = \frac{\frac{1}{z^2+5}}{(z-0)^2} \Rightarrow f(z) = \frac{1}{z^2+5}$$

Noch prüfen: f holomorph in einem Gebiet G mit $\overline{B_2(0)} \subseteq G$?

Nennernst von f: $z_{1/2} = \pm i\sqrt{5}$

wegen $|z_{1/2}| = |\pm i\sqrt{5}| > 2$ ist f holomorph in $G = B_2(0)$

$\Rightarrow$ CIF anwendbar

$$\int_{|z|=2} \frac{1}{z^2(z^2+5)} \, dz = \int_{|z|=2} \frac{f(z)}{(z-0)^2} \, dz.$$

$\underline{k=1}$

$$= 2\pi i f'(0) = 2\pi i \left[\frac{d}{dz} \frac{1}{z^2+5}\right]\Big|_{z=0} = 2\pi i \cdot \frac{-2z}{(z^2+5)^2} = 0$$

b) $$\int_{|z|=3} \frac{e^{z}-e^{-z}}{z^4}\,dz = \frac{2\pi i}{3}$$

Def: $f(z) = e^{z} - e^{-z}$ dann f hol. auf $\mathbb{C}$, also auch in $B_3(0)$

$$\underset{CIF}{\Rightarrow} \int_{|z|=3} \frac{f(z)}{z^4}\,dz = \frac{2\pi i}{3!} f^{(3)}(0) = \frac{\pi i}{3}\left(e^{z}+e^{-z}\right)\Big|_{z=0} = \frac{2\pi i}{3}$$

A19
$$F(z) = \frac{1}{2\pi i} \int_{|y|=2} \frac{e^{z^2 y}}{y^2+1}\,dy, \; z \in \mathbb{C}$$

Finde mittels CIF integralfreie Darstellung von F

suche Nennernst.: $y^2+1 = (y+i)(y-i)$

PZB: $$\frac{1}{y^2+1} = \frac{a}{y+i} + \frac{b}{y-i}$$

$$\Rightarrow \dots \Rightarrow a = \frac{i}{2},\; b = -\frac{i}{2}$$

$$\Rightarrow \frac{1}{y^2+1} = \frac{i}{2}\left(\frac{1}{y+i} - \frac{1}{y-i}\right)$$

Damit:
$$F(z) = \frac{1}{2\pi i}\frac{i}{2}\left[\int_{|y|=2} \frac{e^{z^2 y}}{y+i}\,dy - \int_{|y|=2} \frac{e^{z^2 y}}{y-i}\,dy\right]$$

$$\underset{\substack{\uparrow\\ \text{mit Th 3.8}}}{=} \frac{1}{2\pi i}\frac{i}{2}\left[\int_{|y+i|=2} \frac{e^{z^2 y}}{y+i}\,dy - \int_{|y-i|=2} \frac{e^{z^2 y}}{y-i}\,dy\right]$$

$$\underset{CIF}{=} \frac{i}{2}\left[e^{z^2 y}\Big|_{y=-i} - e^{z^2 y}\Big|_{y=i}\right]$$

$$= \frac{i}{2}\left(e^{-iz^2} - e^{iz^2}\right)$$

$$= \frac{e^{iz^2} - e^{iz^2}}{2i} = \sin(z^2)$$

<u>Taylor</u> (Th 4,2)

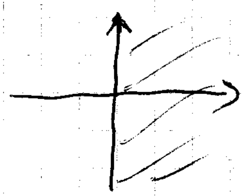
$f: B_R(a) \to \mathbb{C}$ hol. Dann hat f die Entwicklung $f(z) = \sum_{n=0}^{\infty} b_n (z-a)^n$,

$$b_n = \frac{1}{2\pi i} \int_{|z-a|=R} \frac{f(z)}{(z-a)^{n+1}} dz$$

<u>A20</u>

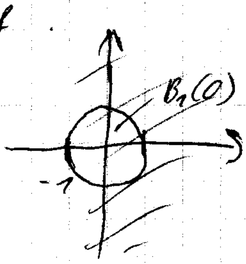
$f(z) = \mathrm{Log}(z+1)$

$\mathrm{Log}(z)$ ist hol. auf $\mathbb{C} \setminus \{x \in \mathbb{R} ; x \leq 0\}$



$\Rightarrow \mathrm{Log}(z+1)$ ist hol. auf

$\mathbb{C} \setminus \{x \in \mathbb{R} ; x \leq -1\}$



und insbes. in

$B_1(0)$.

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} b_n z^n \quad \text{mit } b_n = \frac{1}{2\pi i} \int_{|z|=1} \frac{\mathrm{Log}(z+1)}{z^{n+1}} dz$$

$$b_0 = \frac{1}{2\pi i} \int_{|z|=1} \frac{\mathrm{Log}(z+1)}{z} dz \underset{\uparrow \atop CIF}{=} \mathrm{Log}(z+1)\Big|_{z=0} = 0$$

<u>$n \geq 1$</u>

$$b_n = \frac{1}{2\pi i} \int_{|z|=1} \frac{\mathrm{Log}(z+1)}{z^{(n+1)}} \underset{\uparrow \atop CIF}{=} \frac{1}{n!} \left[\frac{d^n}{dz^n} \mathrm{Log}(z+1) \right]_{z=0} \Big\}$$

Beh.: $\frac{d^n}{dz^n} \mathrm{Log}(z+1) = (-1)^{n-1} (n-1)! \cdot \frac{1}{(z+1)^n}$

$n = 1$: $\frac{d}{dz} \mathrm{Log}(z+1) = \frac{1}{z+1}$ ✓

$n \to n+1$: $\frac{d^{n+1}}{dz^{n+1}} \mathrm{Log}(z+1) = \frac{d}{dz} \left[\frac{d^n}{dz^n} \mathrm{Log}(z+1) \right]$

$$= \frac{e^{iz^2} - e^{iz^2}}{2i} = \sin(z^2)$$

<u>Taylor</u> (Th 4,2)

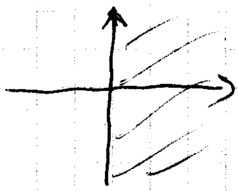
$f: B_R(a) \to \mathbb{C}$ hol. Dann hat f die Entwicklung $f(z) = \sum_{n=0}^{\infty} b_n (z-a)^n$, ~~für~~

$$b_n = \frac{1}{2\pi i} \int_{|z-a|=R} \frac{f(z)}{(z-a)^{n+1}} dz$$

<u>A20</u>

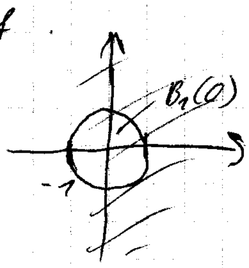
$f(z) = \operatorname{Log}(z+1)$

$\operatorname{Log}(z)$ ist hol. auf $\mathbb{C} \setminus \{x \in \mathbb{R} ; x \leq 0\}$



$\Rightarrow \operatorname{Log}(z+1)$ ist hol. auf

$\mathbb{C} \setminus \{x \in \mathbb{R} ; x \leq -1\}$



und insbes. in

$B_1(0)$.

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} b_n z^n \quad \text{mit } b_n = \frac{1}{2\pi i} \int_{|z|=1} \frac{\operatorname{Log}(z+1)}{z^{n+1}} dz$$

$$b_0 = \frac{1}{2\pi i} \int_{|z|=1} \frac{\operatorname{Log}(z+1)}{z} dz \underset{\text{CIF}}{=} \operatorname{Log}(z+1)\Big|_{z=0} = 0$$

<u>$n \geq 1$</u>

$$b_n = \frac{1}{2\pi i} \int_{|z|=1} \frac{\operatorname{Log}(z+1)}{z^{(n+1)}} \underset{\text{CIF}}{=} \left\{ \frac{1}{n!} \left[\frac{d^n}{dz^n} \operatorname{Log}(z+1) \right]_{z=0} \right\}$$

Beh.: $\frac{d^n}{dz^n} \operatorname{Log}(z+1) = (-1)^{n-1} (n-1)! \cdot \frac{1}{(z+1)^n}$

$n = 1$: $\frac{d}{dz} \operatorname{Log}(z+1) = \frac{1}{z+1}$ ✓

$n \to n+1$: $\frac{d^{n+1}}{dz^{n+1}} \operatorname{Log}(z+1) = \frac{d}{dz} \left[\frac{d^n}{dz^n} \operatorname{Log}(z+1) \right]$

$$\underset{(IV)}{\overset{\uparrow}{=}} \frac{d}{dz}\left[(-1)^{n-1}(n-1)!\,\frac{1}{(z+1)^n}\right] = (-1)^{n-1}(n-1)!\,\frac{-1}{(1+z)^{2n}}\cdot n\cdot(1+z)^{n-1}$$

$$= (-1)^n\, n!\,\frac{1}{(1+z)^{n+1}} \checkmark$$

$$\Rightarrow b_n = \frac{1}{n!}\,(-1)^{n-1}(n-1)!\,\frac{1}{(1+z)^n}\Big|_{z=0}$$

$$= \frac{(-1)^{n-1}}{n} \Rightarrow f(z) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}$$

Konvergenzradius R:

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|b_n|}}$$

$$\sqrt[n]{|b_n|} = \sqrt[n]{\frac{1}{n}} = \frac{1}{\sqrt[n]{n}} \xrightarrow[n\to\infty]{} 1$$

$$\Rightarrow \underline{\underline{R = 1}}$$

<u>A21</u>

Satz von Liouville: $f: \mathbb{C}\to\mathbb{C}$ hol.

$\exists\, M>0$ sodass $|f(z)| \le M \;\forall z\in\mathbb{C}$

$\Rightarrow f$ konstant

$f: \mathbb{C}\to\mathbb{C}$ hol. und es gelte

$Re(f(z)) \le C \;\; \forall z\in\mathbb{C}$

zeige: f konstant

dazu: betrachte $e^{f(z)}$ (hol. auf $\mathbb{C}$).

$$|e^{f(z)}| = |e^{Re(f(z))}|\,\underbrace{|e^{i\,Im(f(z))}|}_{1}$$

$$= e^{Re(f(z))} < e^{C}$$

$\underset{\text{Liouville}}{\Rightarrow} e^{f(z)}$ konstant $\Rightarrow f(z)$ konstant