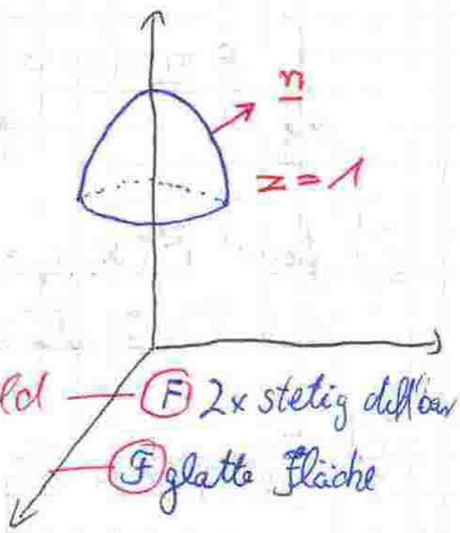
050

3.02.10

$\mathcal{F} = \{(x,y,z) \in \mathbb{R}^3 : z = 2 - x^2 - y^2 \quad x^2+y^2 < 1 \quad z > 0\}$

$F(x,y,z) = (yz, z, 0)$

$I = \int_{\mathcal{F}} \operatorname{rot} F \, \underline{n} \, d\omega = -\pi$




mit dem Satz von Stokes:

$$\int_{\mathcal{F}} \operatorname{rot} F \, \underline{n} \, d\omega = \int_{\partial \mathcal{F}} F \, dx$$

Vektorfeld — F 2x stetig diff'bar

Fläche — $\mathcal{F}$ glatte Fläche

wobei $\partial\mathcal{F} = \Gamma$ die Orientierung trägt, die durch die Rechte-Hand-Regel induziert ist

geeignete Parametrisierung $\gamma(t) = (\cos t, \sin t, 1)$

$\Gamma = \{(x,y,z) \in \mathbb{R}^3 \quad z = 1 \quad x^2+y^2 = 1\}$ (unabh. pos. orientiert)

$= \gamma([0, 2\pi])$

$$\int_{\partial\mathcal{F}} F \, dx = \int_0^{2\pi} F(\gamma(t)) \cdot \dot{\gamma}(t) \, dt = \int_0^{2\pi} \begin{pmatrix} \sin t \cdot 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt$$

$$= \int_0^{2\pi} -\sin^2 t + \cos t \, dt = -\int_0^{2\pi} \sin^2 t \, dt + \underbrace{\int_0^{2\pi} \cos t \, dt}_{=0}$$

Halbwinkelform

$$= -\int_0^{2\pi} \frac{1}{2}(1 - \cos(2t)) \, dt$$

$$= -\frac{1}{2} \int_0^{2\pi} dt + \frac{1}{2} \int_0^{2\pi} \cos(2t) \, dt \qquad s = 2t \quad \frac{ds}{dt} = 2$$

$$dt = \frac{ds}{2}$$

$$= -\frac{2\pi}{2} + \frac{1}{2} \int_0^{2\pi} \frac{\cos(s)}{2} \, ds$$

$= -\pi$

Gauß:

$G = \{(x,y,z) \; x^2+y^2+z^2 < 1\} = B_1(0)$

$$\int_{\partial G} F \, \underline{n} \, d\omega = \int_G \operatorname{div} F \, dx\,dy\,dz$$ wenn gilt G = Gaußgebiet, F = stetig diff'bar

i) für $F(x,y,z) = (yz, z, 0)$ ist $I = 0$ da $\operatorname{div} F(x,y,z) = 0$ //

Gauß-Gebiet: Rand aus endl. vielen Platten/Flächenstücken

ii) $F(x,y,z) = (0, yz, z)$

berechne $\int_{\partial G} F \, \underline{n} \, d\omega$ (zu Fuß) | ohne Gauß | hier: $\underline{n}$ ist Ortsvektor $= \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$= \int_{\partial G} \begin{pmatrix} 0 \\ yz \\ z \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} d\omega = \int_{\partial G} y^2 z + z^2 \, d\omega \qquad d\omega = \sqrt{EG - F} \, d\varphi \, d\theta$$

sphärische Polarkoordinaten $\underline{X}(\theta, \varphi) = \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix}$

$$E = \left(\frac{\partial x}{\partial \theta}\right)^2 = 1 \qquad F = \frac{\partial x}{\partial \theta} \cdot \frac{\partial x}{\partial \varphi} = 0 \qquad G = \sin^2 \varphi$$

$$\Rightarrow d\omega = \sin\theta \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} \int_0^{\pi} [\sin^2\theta \sin^2\varphi \cos\theta) + \cos^2\theta] \sin\theta \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} \int_0^{\pi} \sin^3\theta \cos\theta \sin^2\varphi + \cos^2\theta \sin\theta \, d\theta \, d\varphi$$

$$= \underbrace{\int_0^{2\pi} \int_0^{\pi} \frac{d}{d\theta} \left(\frac{\sin^4\theta}{4}\right) d\theta \, d\varphi \, \sin^2\varphi}_{0} + \underbrace{\int_0^{2\pi} \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta \, d\varphi}_{-2\pi \int_0^{\pi} \frac{d}{d\theta} \left(\frac{\cos^3\theta}{3}\right)^0 d\theta}$$

$$= -\frac{2\pi}{3} \cdot [\cos^3\theta]_0^{\pi} = -\frac{2\pi}{3}(-1-1) = \frac{4}{3}\pi$$

mit Gauß

$$\operatorname{div} F(x,y,z) = \frac{\partial(yz)}{\partial y} + \frac{\partial z}{\partial z} = z+1$$

$$\underline{I} = \int_G F\,\underline{n}\,d\omega = \int_G \operatorname{div} F\,dx\,dy\,dz = \int_G (z+1)\,dx\,dy\,dz$$

$(G = B_1(0))$ $$= \underbrace{\int_{B_1(0)} z\,dx\,dy\,dz}_{\text{Polarkoord.}} + \underbrace{\int_{(B_1(0))} 1\,dx\,dy\,dz}_{= B_1(0) = \frac{4\pi}{3}}$$

mit dem Cavalieri-Prinzip

hier ist der Radius 1

$$\int_0^1\int_0^{2\pi}\int_0^{\pi} (r\cos\theta)\,\overbrace{r^2\sin\theta}^{\text{Funktionaldeterminante im } \mathbb{R}^3}\,d\theta\,d\varphi\,dr$$

$$= \int_0^1 r^3 \int_0^{2\pi} \Big(\int_0^{\pi} \underbrace{\cos\theta\sin\theta}_{\frac{d}{d\theta}\left(\frac{\sin^2\theta}{2}\right)}\,d\theta\Big)\,d\varphi\,dr$$

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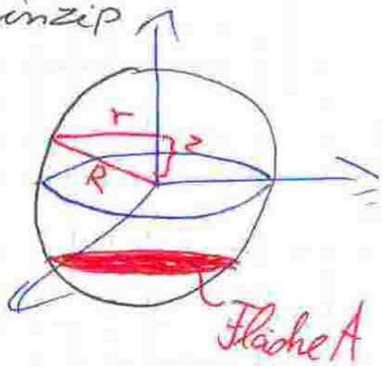
Berechnung nach dem Cavalieri-Prinzip




Was ist $A(z)$? $= \pi r^2$ $\quad R^2 = r^2 + z^2$

$$A(z) = \pi r^2 = \pi(R^2 - z^2)$$

$$B_R(0) = V = \int_{-R}^{R} \pi(R^2 - z^2)\,dz$$

Radius

$$= \pi R^2 \int_{-R}^{R} dz - \pi\int_{-R}^{R} z^2\,dz$$

$$= 2\pi R^3 - \pi\left[\frac{z^3}{3}\right]_{-R}^{R}$$

$$= 2\pi R^3 - \frac{\pi}{3}(R^3 + R^3) = 2\pi R^3 - \frac{2R^3\pi}{3}$$

$$= \frac{6}{3}\pi R^3 - \frac{2}{3}\pi R^3 = \underline{\underline{\frac{4}{3}\pi R^3}}$$