Transformationsformel

<u>Satz:</u> $\psi = (\xi, \eta) : \bar{G}^* \longrightarrow \bar{G}$ umkehrbar eindeutig und diff'bar mit

$$\det \nabla\psi(s,t) = \frac{\partial(\xi,\eta)}{\partial(s,t)} = \det \begin{pmatrix} \frac{\partial \xi}{\partial s} & \frac{\partial \eta}{\partial s} \\ \frac{\partial \xi}{\partial t} & \frac{\partial \eta}{\partial t} \end{pmatrix} \neq 0$$

Dann gilt:

$$\int_G f(x,y)\,dx\,dy = \int_{G^*} f(\xi(s,t), \eta(s,t)) \left| \frac{\partial(\xi,\eta)}{\partial(s,t)} \right| ds\,dt$$

Polarkoordinaten: $\left.\begin{array}{l} \xi(r,\varphi) = r\cos\varphi \\ \eta(r,\varphi) = r\sin\varphi \end{array}\right\} \frac{\partial(\xi,\eta)}{\partial(r,\varphi)} = r$

<u>Satz:</u> $\psi = (\xi, \eta, \zeta) : \underset{\mathbb{R}^3}{\underset{\cap}{\bar{G}^*}} \longrightarrow \underset{\mathbb{R}^3}{\underset{\cap}{\bar{G}}}$ umkehrbar eindeutig und diff'bar mit

~~det~~

$$\det \nabla\psi(r,s,t) = \frac{\partial(\xi,\eta,\zeta)}{\partial(r,s,t)} = \det \begin{pmatrix} \frac{\partial \xi}{\partial r} & \frac{\partial \eta}{\partial s} & \vdots \\ \frac{\partial \xi}{\partial s} & \vdots & \vdots \\ \frac{\partial \xi}{\partial t} & \vdots & \vdots \end{pmatrix} \neq 0$$

Dann gilt:

$$\int_G f(x,y,z)\,dx\,dy\,dz = \int_{G^*} f(\xi(r,s,t), \eta(r,s,t), \zeta(r,s,t)) \left| \frac{\partial(\xi,\eta,\zeta)}{\partial(r,s,t)} \right| dr\,ds\,dt$$

<u>Beispiel:</u>

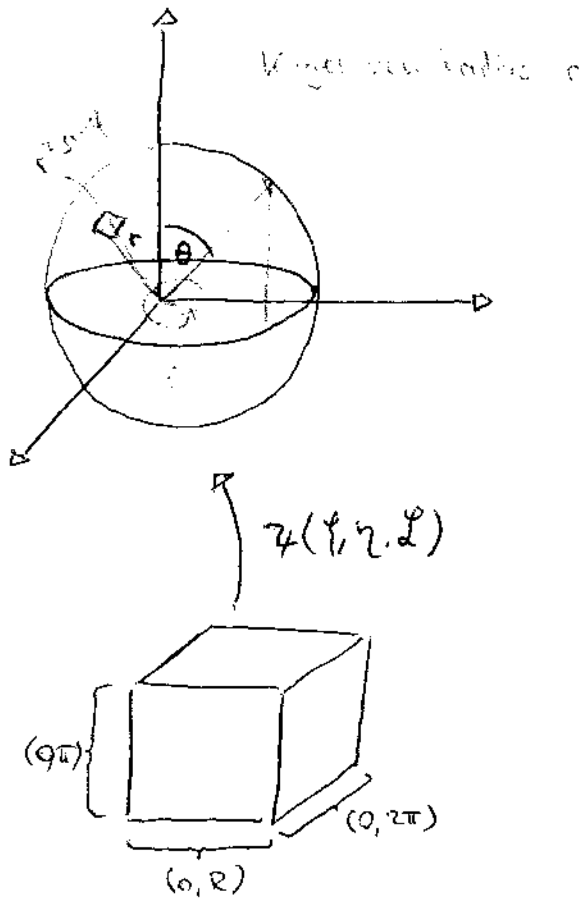
"Was ist das Volumen einer Kugel vom Radius R?"

$$\int_{B_R(0)} 1\,dx\,dy\,dz = \mathrm{Vol}(B_R(0))$$

$$B_R(0) = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < R^2\}$$

Kugelkoordinaten:

$0 < r < R$
$0 < \varphi < 2\pi$
$0 < \theta < \pi$



$$\xi(r,\varphi,\theta) = r \sin\theta \cos\varphi$$
$$\eta(r,\varphi,\theta) = r \sin\theta \sin\varphi$$
$$\zeta(r,\varphi,\theta) = r \cos\theta$$

$\theta = 0:$
$$\xi(r,\varphi,0) = 0$$
$$\eta(r,\varphi,0) = 0$$
$$\zeta(r,\varphi,0) = r$$

$$\frac{\partial(\xi,\eta,\zeta)}{\partial(r,\varphi,\theta)} = \det\begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ -r\sin\theta\sin\varphi & r\sin\theta\cos\varphi & 0 \\ r\cos\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\theta \end{pmatrix}$$

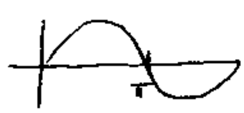
$$= r^2\left(\cos\theta\left(-\sin\theta\cos\theta\sin^2\varphi - \cos\theta\sin\theta\cos^2\varphi\right)\right)$$
$$+ r^2\left(-\sin\theta\left(\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi\right)\right)$$
$$= -r^2\left(\cos^2\theta\left(\sin\theta\sin^2\varphi + \sin\theta\cos^2\varphi\right)\right)$$
$$-r^2\left(\sin^2\theta\left(\sin\theta\cos^2\varphi + \sin\theta\sin^2\varphi\right)\right)$$

$\boxed{\sin^2\varphi + \cos^2\varphi = 1}$

$$= -r^2\left\{\cos^2\theta\sin\theta\underbrace{(\sin^2\varphi+\cos^2\varphi)}_{=1} + \sin^2\theta\sin\theta\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{=1}\right\}$$

$$= -r^2\left\{\sin\theta\left(\cos^2\theta + \sin^2\theta\right)\right\} = -r^2\sin\theta$$

$$\frac{\partial(\xi,\eta,\zeta)}{\partial(r,s,t)} = -r^2\sin\theta$$



$\theta \in (0,\pi)$

$r, \varphi, \theta \rightsquigarrow r, \theta, \varphi$

$$\frac{\partial(\xi,\eta,\zeta)}{\partial(r,\theta,\varphi)} = r^2 \sin\theta \qquad \geq 0 \quad \theta\in[0,\pi]$$

$$\Longrightarrow \left|\frac{\partial(\xi,\eta,\zeta)}{\partial(r,s,t)}\right| = r^2 \sin\theta$$

Ebene Polarkoordinaten entsprechen

$\vartheta = \frac{\pi}{2} \rightsquigarrow \sin\frac{\pi}{2} = 1$

$$\int_{B_R(0)} 1 \, dx\, dy\, dz = \int_0^R\int_0^\pi\int_0^{2\pi} \boxed{r^2 \sin\theta}\, d\varphi\, d\theta\, dr$$

$$= 2\pi \int_0^R\int_0^\pi r^2 \sin\theta \, d\theta\, dr$$

$$= 2\pi \left(\int_0^R r^2\, dr\right)\left(\int_0^\pi \sin\theta\, d\theta\right)$$

$$= 2\pi \left[\frac{1}{3}r^3\right]_0^R \left[-\cos\theta\right]_0^\pi$$

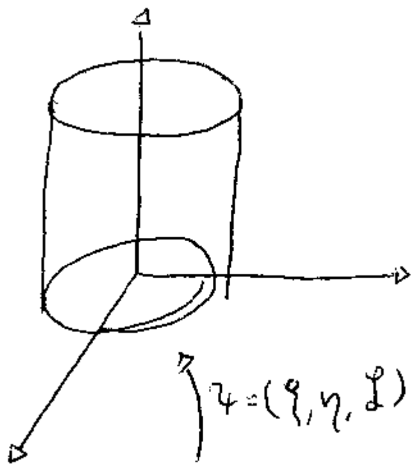
$$= 2\pi \,\frac{1}{3}R^3\,(1-(-1)) = 2\pi\,\frac{1}{3}R^3\cdot 2 = \frac{4}{3}\pi R^3$$

Zylinderkoordinaten

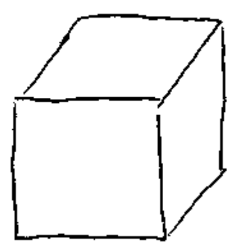
$$\xi = r\cos\varphi$$

$$\eta = r\sin\varphi$$

$$\zeta = z$$



$$\frac{\partial(\xi,\eta,\zeta)}{\partial(r,\varphi,z)} = \det\begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -r\sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$= 1\,(r\cos^2\varphi + r\sin^2\varphi)$$

$$= \underline{r}$$

SATZ ÜBER IMPLIZITE FUNKTIONEN

Sei $F : G \to \mathbb{R}^2$, $G \subset \mathbb{R}^2$

Wann invertierbar?

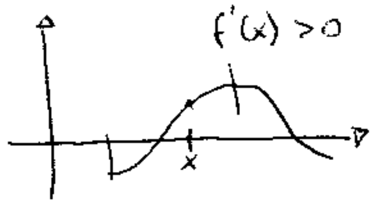
$$\begin{tikzcd} G \arrow[r, "F"] & F(G) \arrow[l, bend left, "F^{-1}"] \end{tikzcd}$$

Lineare Funktionen:

$F(\underline{v}) = A\underline{v}$ $\quad G = \mathbb{R}^2 \quad \underline{v} \in \mathbb{R}^2; \; A \in \mathbb{R}^{2\times 2}$

Invertierbar $\Longleftrightarrow$ $\det A \neq 0$

allgemein? ... kann man nur von "lokaler Invertierbarkeit" sprechen.



Satz: $F : G \to \mathbb{R}^2$ ist lokal bei (x_0, y_0) invertierbar, falls

F stetig diff'bar.