$\int_\Gamma f\,d\underline{x}$ unabhängig vom Weg (d.h. f konservativ in $\mathcal{G}$)

$\iff \oint_{\Gamma^*} f\,d\underline{x} = 0 \quad \forall\, \Gamma^* \subset \mathcal{G}$ geschlossen

(1. HS für Kurvenintegrale)

$\iff f = \nabla h$

"$\Longleftarrow$" $\int_\Gamma f\,d\underline{x} = \int_\Gamma \nabla h(\underline{x})\,dx$

(Kettenregel) u. HS. Int./Diff. Rechnung $= h(Q) - h(P)$

$= 0$ falls $P = Q$

also für geschl. Kurven.

"$\Longrightarrow$" schwierig

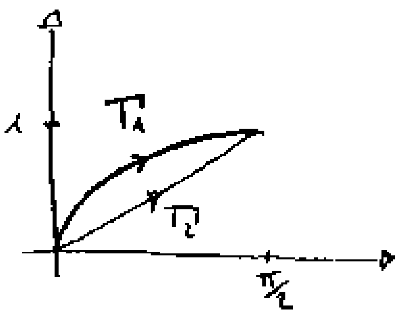
Idee: def.: $h(\underline{x}) = \int_{\Gamma(x_0,\underline{x})} f\,dx$ wohl definiert

und $\nabla h(\underline{x}) = f$

Beispiel: $f(x,y) = (y, -x)$

betr. $\Gamma_1 = \{\underbrace{(t, \sin t)}_{=\gamma(t)},\ t \in [0, \pi/2]\}$

$\Gamma_2 = \{\underbrace{(t, \tfrac{2}{\pi}t)}_{=\tilde{\gamma}(t)},\ t \in [0, \pi/2]\}$



$$\int_{\Gamma_1} f\,d\underline{x} = \int_0^{\pi/2} f(\gamma(t))\cdot\dot{\gamma}(t)\,dt = \int_0^{\pi/2} f_1(\gamma(t))\,\dot{\gamma}_1(t) + f_2(\gamma(t))\,\dot{\gamma}_2(t)\,dt$$

$$= \int_0^{\pi/2} \sin(t)\cdot 1 + (-t)\cos(t)\,dt$$

$$= [-\cos t]_0^{\pi/2} - \int_0^{\pi/2} t\cos(t)\,dt$$

$$= 1 - \int_0^{\pi/2} t \frac{d}{dt} \sin t \, dt$$

$$= 1 + \int_0^{\pi/2} 1 \sin t \; dt - [t \sin t]_0^{\pi/2}$$

$$= 2 - \frac{\pi}{2}$$

$$\int_{T_2} \underline{f} \, d\underline{x} = \int_0^{\pi/2} f_1(\tilde{\gamma}(t))\, \dot{\tilde{\gamma}}_1(t) + f_2(\tilde{\gamma}(t))\, \dot{\tilde{\gamma}}_2(t) \, dt$$

$$= \int_0^{\pi/2} \underbrace{\frac{2}{\pi} t \cdot 1 + (-t) \frac{2}{\pi}}_{=0} \, dt = 0$$

also: $\int_{T_1} \underline{f} \, d\underline{x} \neq \int_{T_2} \underline{f} \, d\underline{x}$ obwohl gleicher Anfangs- und Endpunkt!

Übungsaufgabe:

$\underline{g}(x,y) = (x, y)$

dann ist $\underline{g}$ ein Gradientenfeld $\underline{g} = \nabla h$

$h(x,y) = \frac{1}{2}(x^2 + y^2)$

dann muss gelten: $\int_{T_1} \underline{g} \, d\underline{x} = \int_{T_2} \underline{g} \, d\underline{x}$

$$\int_{T_1} \underline{g} \, d\underline{x} = \int_0^{\pi/2} g_1(\gamma(t))\, \dot{\gamma}_1(t) + g_2(\gamma(t))\, \dot{\gamma}_2(t) \, dt$$

$$= \int_0^{\pi/2} t \cdot 1 + \sin t \cos t \, dt$$

$$= \left[\frac{1}{2} t^2\right]_0^{\pi/2} + \int_0^{\pi/2} \underbrace{\sin t \cos t}_{\frac{d}{dt}\left(\frac{1}{2}\sin^2 t\right)} \, dt$$

$$[\ldots] = \left[\tfrac{1}{2} t^2\right]_0^{\pi/2} + \left[\tfrac{1}{2}\sin^2 t\right]_0^{\pi/2} = \frac{1}{2}\,\frac{\pi^2}{4} + \frac{1}{2} = \underline{\underline{\frac{\pi^2}{8} + \frac{1}{2}}}$$

$$\int_{T_1} g \, d\underline{x} = \int_0^{\pi/2} g_1(\tilde{\gamma}(t))\,\dot{\tilde{\gamma}}_1(t) + g_2(\tilde{\gamma}(t))\,\dot{\tilde{\gamma}}_2(t)\, dt$$

$$= \int_0^{\pi/2} t \cdot 1 + \left(\frac{2}{\pi} t\right)\frac{2}{\pi} \; dt$$

$$= \left(1 + \frac{4}{\pi^2}\right) \int_0^{\pi/2} t \; dt = \left(1 + \frac{4}{\pi^2}\right)\left[\tfrac{1}{2} t^2\right]_0^{\pi/2}$$

$$= \left(1 + \frac{4}{\pi^2}\right)\frac{1}{2}\,\frac{\pi^2}{4} = \underline{\underline{\frac{\pi^2}{8} + \frac{1}{2}}}$$

Gauß'scher Integralsatz in der Ebene

1D: $G = (a,b)$; $\partial G = \{a, b\}$

Hauptsatz der Diff | Int Rechnung

$$\underbrace{\int_a^b \frac{d}{dx} F(x)\, dx}_{\text{Gebietsintegral über eine Ableitung}} = \underbrace{F(b) - F(a)}_{\text{„Randintegral über die Funktion"}}$$

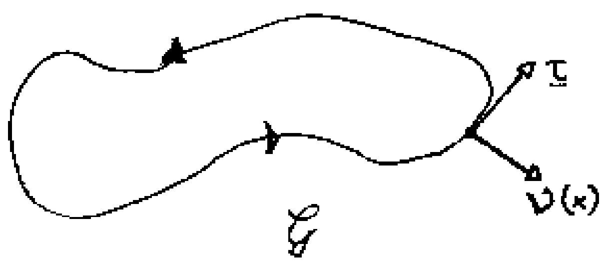
2D:

$G \subset \mathbb{R}^2$ Gebiet mit glattem Rand

$F: \bar{G} \to \mathbb{R}^2$ stetig; F diff'bar in G

dann gilt:

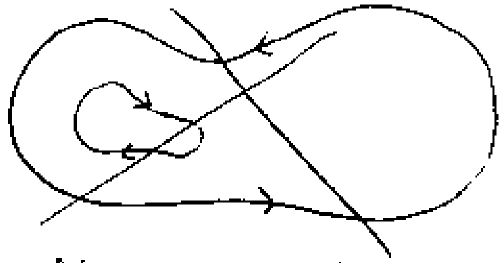
$$\int_G \operatorname{div} F(x)\, dx = \oint_{\partial G} F\nu \; ds$$



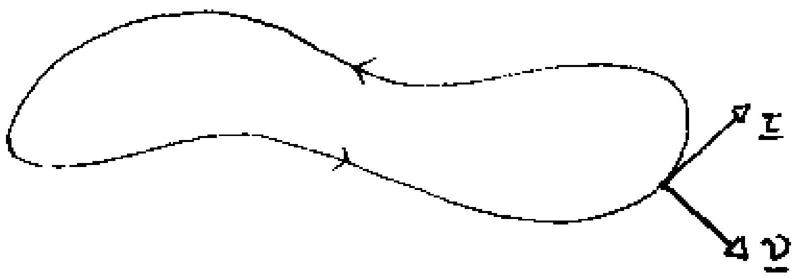
$$\operatorname{div} F = \frac{\partial a}{\partial x}(x,y) + \frac{\partial b}{\partial y}(x,y)$$

$$F(x) = (a(x,y), b(x,y))$$

$\nu(x)$ äußeres Einheitsnormalen Feld
$\tau(x)$ Einheitstangentialfeld



Wir betrachten sog. "einfach zusammenhängende Gebiete"



Wie kann man den Normalenvektor ausdrücken?

$\underline{\nu}(\nu_1, \nu_2)$ $\qquad$ $\underline{\nu}(\tau_2, -\tau_1)$

$\underline{\tau}(\tau_1, \tau_2)$ $\qquad$ $\underline{\tau}(\ ,\ )$

Sei $\partial \mathcal{G} = \Gamma$ geschlossen und gegeben durch eine Kurve $\gamma = \gamma(t)$. Dann gilt

$\tau(\gamma(t)) = \dfrac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}$ kommt in unserer Def. des Kurvenintegrals vor:

$$\left(\int_\Gamma f\, d\underline{x} = \int_0^1 f(\gamma(t))\, \dot{\gamma}(t)\, dt \right)$$

betr.: $\displaystyle\oint_\Gamma \underline{F} \cdot \nu \, ds$

$ds = \|\dot{\gamma}(t)\|\, dt$

$\underline{F}(x,y) = (a(x,y), b(x,y))$

$$\begin{aligned}
\underline{F} \cdot \underline{\nu} &= F_1 \cdot \nu_1 + F_2 \cdot \nu_2 \\
&= F_1 \cdot \tau_2 - F_2\, \tau_1 \\
&= F_1 \cdot \frac{\dot{\gamma}_2(t)}{\|\dot{\gamma}\|} - F_2 \frac{\dot{\gamma}_1}{\|\dot{\gamma}\|} \\
&= (a\, \dot{\gamma}_2 - b\, \dot{\gamma}_1) \frac{1}{\|\dot{\gamma}\|}
\end{aligned}$$

$\rightsquigarrow$

$$\int_{\partial \mathcal{G}} \underline{F} \cdot \underline{\nu} \cdot ds = \int_0^1 \left(a(\gamma(t))\, \dot{\gamma}_2(t) - b(\gamma(t))\, \dot{\gamma}_1(t)\right) \frac{1}{\|\dot{\gamma}(t)\|} \cdot \|\dot{\gamma}(t)\|\, dt$$

[...]

$$= \int_{\Gamma} a\,dy - b\,dx$$

$\Gamma = \partial \mathcal{G}$

$$\boxed{\int_{\mathcal{G}} \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} \right) dx\,dy = \int_{\partial \mathcal{G}} a\,dy - b\,dx}$$

Stokes betr. $F(x,y) = (A(x,y), B(x,y))$

$$\int_{\partial \mathcal{G}} A\,dx + B\,dy = \int_{\partial \mathcal{G}} B\,dy - (-A)\,dx$$

$$= \int_{\mathcal{G}} \underbrace{\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right)}_{=:\ \mathrm{rot}\, F} dx\,dy \quad \text{(Gauß)}$$

Die Rotation in 2D ist skalar: $F = (A, B)$ gilt:

$$\mathrm{rot}\ F = \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}$$

Satz von Stokes:

$$\int_{\mathcal{G}} \mathrm{rot}\ F \ dx\,dy = \int_{\partial \mathcal{G}} A\,dx + B\,dy \ ; \quad F(A,B)$$

$\equiv$ Gauß