HöMa3 Übung4 am 4.11.09

Wiederholung

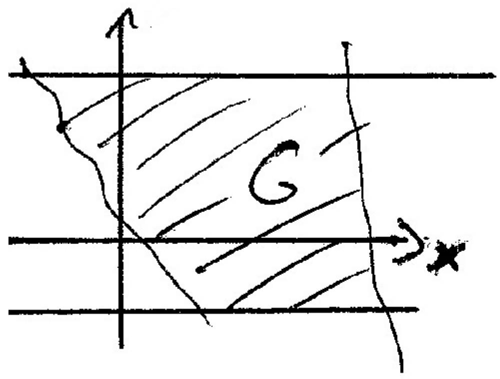
Ebene Gebietsintegrale können wir mit Hilfe von Satz 5.10 berechnen, wenn das Gebiet G von der speziellen Form:

$G = \left\{(x,y) \in \mathbb{R}^2 \middle| \, c < y < d \, , \, a(y) < x < b(y)\right\}$

für stetige Funktionen a,b : $[c,d] \to \mathbb{R}$

Dann gilt für $\bar{G} \to \mathbb{R}$ stetig:

$\int\limits_G f\,(x,y)\,dx\,dy \overset{5.10}{=} \int\limits_c^d \int\limits_{a(y)}^{b(y)} f(x,y)dx\,dy$



A11) $G = \left\{(x,y) \in \mathbb{R}^2 \middle| \, 1 < y < 2 \, , \, y < x < 4-y\right\}$

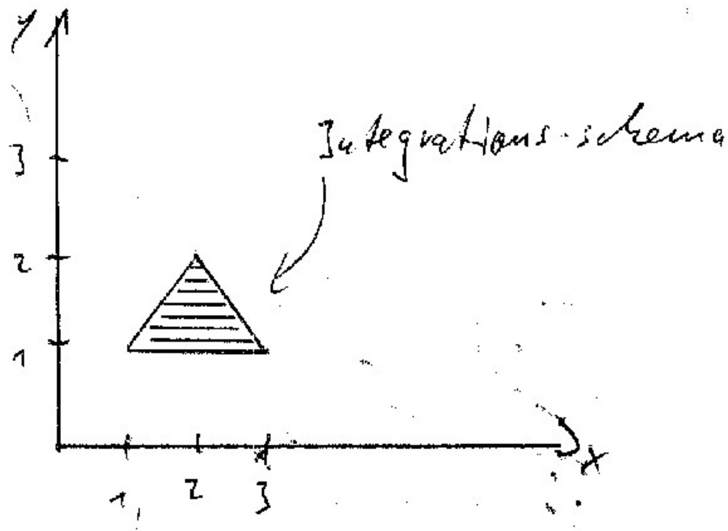
Berechne : $\int\limits_G xy\,dx\,dy = \int\limits_1^2 \int\limits_y^{4-y} xy\,dx\,dy$

$c = 1 \, , \, d = 2 \, , \, a(y) = y \, , \, b(y) = 4-y \, , \, f(x,y) = xy$ ist stetig

$$\int\limits_1^2 \left[\frac{1}{2}y \cdot x^2\right]_y^{4-y} dy = \int\limits_1^2 \left[\frac{1}{2}\left(4-y\right)^2 y - \frac{1}{2}y^3\right] dy = \int\limits_1^2 \frac{1}{2}\left(16-8y+y^2\right)y - \frac{1}{2}y^3\,dy$$

$$= \int\limits_1^2 \left[8y - 4y^2\right] dy = \left[4y^2 - \frac{4}{3}y^3\right]_1^2 = 4 \cdot 2^2 - \frac{4}{3} \cdot 2^3 - 4 + \frac{4}{3}$$

$$= 16 - \frac{32}{3} - 4 + \frac{4}{3} = 2\frac{2}{3}$$



A12) $E := \left\{ (x,y) \in \mathbb{R}^2 \middle| \, \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leq 1 \right\}$

Wegen $\dfrac{y^2}{b^2} \geq 0$ ist $1 \geq \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \geq \dfrac{x^2}{a^2}$ also $x^2 \leq a^2 \Rightarrow |x| \leq a$

Für festes x mit $-a \leq x \leq a$ ist $\dfrac{y^2}{b^2} \leq 1 - \dfrac{x^2}{a^2}$ bzw $|y| \leq b \cdot \sqrt{1 - \dfrac{x^2}{a^2}}$

Den Flächeninhalt der Elipse erhalten wir durch:

$$F_E = \int\limits_E 1 \, dx \, dy = \int\limits_{x=-a}^{a} \int\limits_{y=-b\sqrt{1-\frac{x^2}{a^2}}}^{+b\sqrt{1-\frac{x^2}{a^2}}} 1 \, dy \, dx$$

$$= 2b \cdot \int\limits_{-a}^{a} \sqrt{1 - \frac{x^2}{a^2}} \, dx \overset{\sin t := \frac{x}{a}}{=} 2b \cdot \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\sqrt{1 - \sin^2 t}}_{\cos t} \cdot a \cdot \cos t \text{ dt}$$

$$= 2ab \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \, dt = 2ab \cdot \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos^2 t}{2} \, dt = ab \left[t + \frac{1}{2} \sin 2t \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= ab \cdot \left(\frac{\pi}{2} + \frac{1}{2} \left(\sin \pi - \sin(-\pi) \right) \right) = a \cdot b \cdot \pi$$

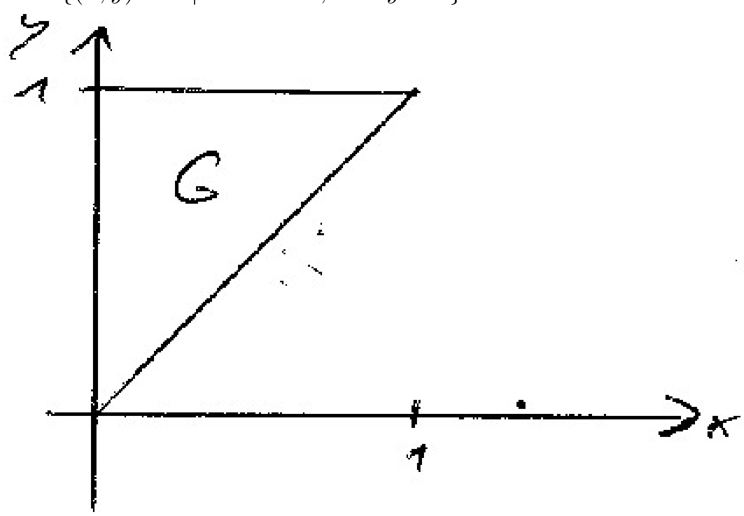
A13) $F : [0,1] \to \mathbb{R}\;\; F(x) := \int\limits_x^1 y \cdot \sin \dfrac{x}{y}\, dy$

Berechne: $\int\limits_0^1 F(x)\, dx \; =: \; I$.

$I = \int\limits_0^1 \int\limits_x^1 y \cdot \sin\left(\dfrac{x}{y}\right) dy\, dx = \int\limits_G f(x,y)\, dy\, dx$

$f\,(x,y) = y \cdot \sin\left(\dfrac{x}{y}\right)$ und

$G = \{(x,y) \in \mathbb{R} \,|\, 0 < x < 1\,,\; x < y < 1\}$



Satz 5.10 ist anwendbar, da $f\,(x,y)$stetig fortsetzbar auf $\bar{G}$ ist.

$\Rightarrow I = \int\limits_{y=0}^{1} \int\limits_{x=0}^{y} f\,(x,y) \quad \underbrace{dx\, dy}_{vertauscht}$

Somit ist die Integrationsreihenfolge vertauscht.

$= \int\limits_{y=0}^{1} \int\limits_{x=0}^{y} y \cdot \sin\left(\dfrac{x}{y}\right) dx\, dy = \int\limits_{y=0}^{1} \left[y \left(-\cos\left(\dfrac{x}{y}\right)\right) y \right]_0^y dy$

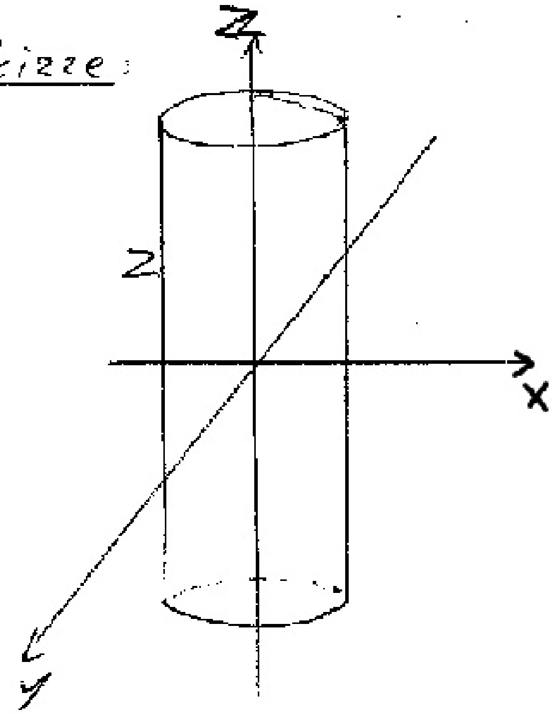
$= -\int\limits_0^1 y^2 \left(\cos(1) - \cos(0)\right) dy = -\left(\cos(1) - 1\right) \int\limits_0^1 y^2$

$= \dfrac{1}{3}\left(1 - \cos(1)\right)$

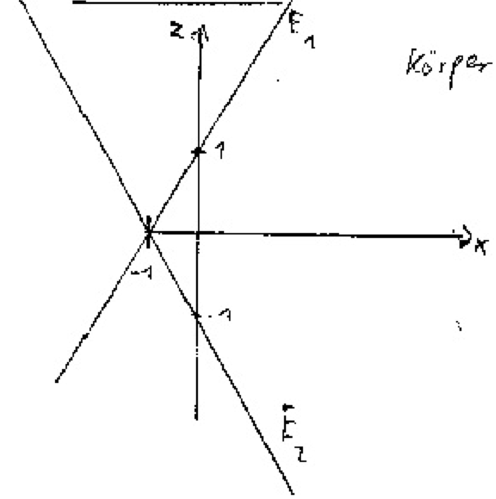
A14)

$E_1 := \{(x, y, z) \in \mathbb{R}^3 \mid z = 1 + x\}$
$E_2 := \{(x, y, z) \in \mathbb{R}^3 \mid z = -(1 + x))\}$
$Z_z := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$

Skizze:



Seitenansicht



Zu Untersuchen gilt der :
Körper $K = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 < 1\,,\, -(1 + x) < z < 1 + x\}$
$x^2 + y^2 < 1 \Leftrightarrow |y| < \sqrt{1 - x^2}$, also :
$-\sqrt{1 - x^2} < y < \sqrt{1 - x^2}$
Zusammen:
$K = \{(x, y, z) \in \mathbb{R}^3 \mid -1 < x < 1\,,\, -\sqrt{1 - x^2} < y < \sqrt{1 - x^2}\,,\, -(1 + x) < z < 1 + x\}$

$$V_k = \int_K 1\, dz\, dy\, dx$$
$$= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=-(1+x)}^{1+x} 1\, dz\, dy\, dx$$
$$= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2 + 2x\, dy\, dx$$
$$= \int_{-1}^{1} (2 + 2x) \cdot 2\sqrt{1 - x^2} dx$$

$$
\begin{aligned}
&= 4 \cdot \int_{-1}^{1} \sqrt{1-x^2}dx + 4\int_{-1}^{1} x \cdot \sqrt{1-x^2}dx \\
&= 8\int_{0}^{1} \sqrt{1-x^2}dx + \underbrace{4 \cdot \left[-\frac{1}{3}\left(1-x^2\right)^{3/2}\right]_{-1}^{1}}_{0} \\
&\overset{x:=\sin t}{=} \ 8 \cdot \int_{0}^{\frac{\pi}{2}} \underbrace{\sqrt{1-\sin^2 t}}_{\cos t} \cdot \cos t \, dt \\
&= 8 \cdot \int_{0}^{\frac{\pi}{2}} \cos^2 t \, dt \ = 8 \cdot \int_{0}^{\frac{\pi}{2}} \frac{1+\cos 2t}{2} \, dt \\
&= 4 \cdot \left[t + \tfrac{1}{2}\sin 2t\right]_{0}^{\frac{\pi}{2}} \ = 4 \cdot \tfrac{\pi}{2} + \frac{\sin \pi}{2} = 2\pi
\end{aligned}
$$