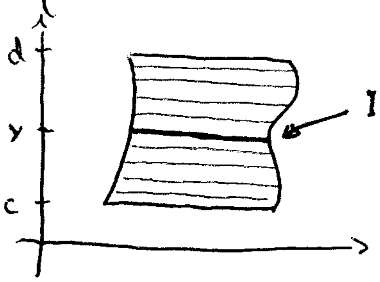
Berechnung von Flächeninhalten und Volumina nach dem Cavalieri Prinzip

2D: $\mathcal{G} = \{(x,y) \in \mathbb{R}^2 : a(y) < x < b(y);\ c < y < d\}$



$I(y) = (a(y), b(y))$

$|I(y)| = b(y) - a(y)$

Cavalieri: $|\mathcal{G}| = \int_c^d |I(y)|\, dy = \int_c^d (b(y) - a(y))\, dy$

3D: $\mathcal{G} = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in g(z);\ z \in (c,d)\}$

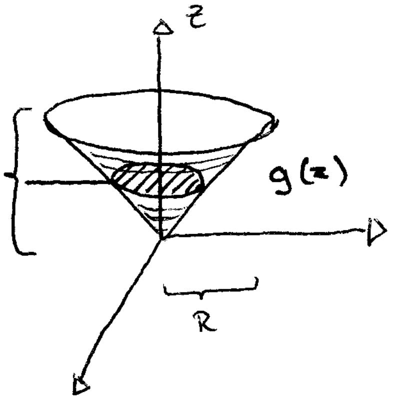
Gebiet (entspricht $I(y)$ in 2D)

Cavalieri: $|\mathcal{G}| = \int_c^d |g(z)|\, dz$

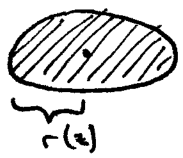
Beispiele:

1\. Volumen eines ~~Zylinders~~ Kegels:

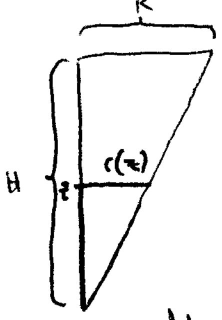
$z - H$



$|g(z)| = \pi\, r(z)^2$



Was ist r(z)?



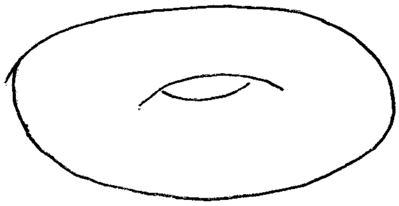
Dreiecksatz: $\frac{R}{H} = \frac{r(z)}{z}$

d.h. $r(z) = \frac{R}{H} \cdot z$

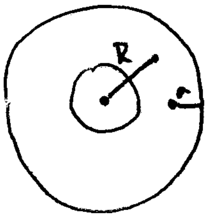
$$V = \int_0^H |g(z)|\, dz = \int_0^H \pi \left(\frac{R}{H} z\right)^2 dz$$

$$= \pi \frac{R^2}{H^2} \int_0^H z^2\, dz = \frac{\pi}{3} \frac{R^2}{H^2} H^3 = \underline{\underline{\frac{\pi R^2 H}{3}}}$$

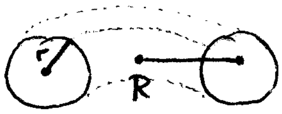
2. Volumen eines Torus:

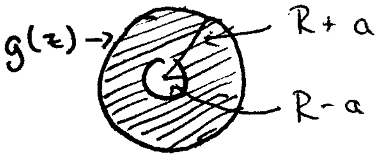


Draufschau:

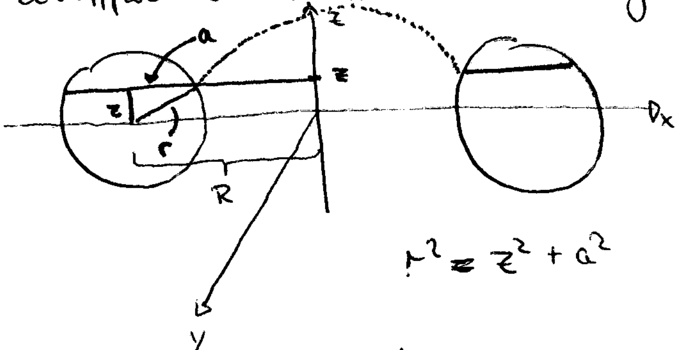


Querschnitt:





Als Schnittflächen erhalten wir Ringe



$r^2 = z^2 + a^2 \quad$ d.h. $\quad a = \sqrt{z^2 - r^2}$

$$|g(z)| = \pi\left((R+a)^2 - (R-a)^2\right)$$
$$= \pi\left(R^2 + a^2 + 2ar - R^2 - a^2 + 2ar\right)$$
$$= 4\pi R \cdot a(z)$$
$$= 4\pi R\sqrt{r^2 - z^2}$$

Cavalieri: Volumen des halben Torus

$$\frac{1}{2} V = \int_0^r |g(z)|\, dz$$

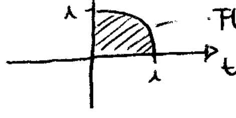
$$\Rightarrow V = 2 \int_0^r 4\pi R \sqrt{r^2 - z^2}\, dz = 8\pi R \int_0^r \sqrt{r^2 - z^2}\, dz$$

$$\int_0^r \sqrt{r^2 - z^2}\, dz = \int_0^r \sqrt{r^2\left(1 - \left(\frac{z}{r}\right)^2\right)}\, dz = r \int_0^r \sqrt{1 - \left(\frac{z}{r}\right)^2}\, dz$$

Substitution:
$t = z/r$

$\frac{dz}{dt} = r$

$$= r^2 \int_0^1 \underbrace{\sqrt{1 - t^2}}\, dt = r^2 \cdot \frac{1}{4}(\pi) = r^2 \frac{\pi}{4}$$



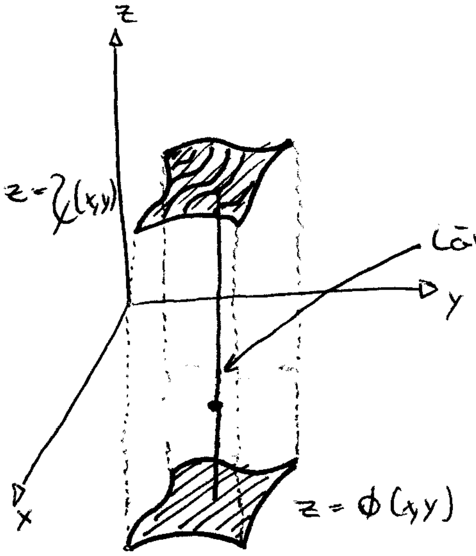
Flächeninhalt $\frac{\pi}{4}$

[...]

$$V = 8\pi R \cdot r^2 \frac{\pi}{4} = 2\pi^2 R r^2$$

Andere Darstellung von Gebieten in $\mathbb{R}^3$

$$\mathcal{G} = \{ (x,y,z) \in \mathbb{R}^3 : \phi(x,y) < z < \psi(x,y),\ (x,y) \in g \}$$



Länge $= \psi(x,y) - \phi(x,y) =$

$$\text{Volumen} = \int_g |I(x,y)| \, dx\,dy$$

Satz:

$$\mathcal{G} = \{ (x,y) \in \mathbb{R}^2 : a(y) < x < b(y);\ y \in (c,d) \} \qquad (a,b \text{ stetig})$$

$f : \mathcal{G} \to \mathbb{R}$ stetig. Dann gilt:

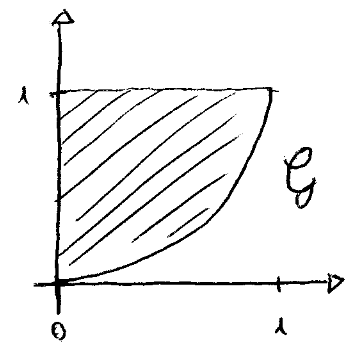
$$\int_{\mathcal{G}} f(x,y)\, dx\,dy = \int_c^d \left(\int_{a(y)}^{b(y)} f(x,y)\, dx \right) dy$$

Beachte: $f(x,y) = 1 \longleftrightarrow$ Cavalieri Prinzip !

Beispiel:

$$f(x,y) = xy; \quad \mathcal{G} = \{ (x,y) \in \mathbb{R}^2 : 0 < x < 1;\ x^3 < y < 1 \}$$

Vertausche Rollen von x und y im Satz:



$$\int_G x\cdot y \; dx\, dy = \int_0^1 \Big(\int_{x^3}^1 x y \, dy \Big) dx$$

$$= \int_0^1 x \underbrace{\Big(\int_{x^3}^1 y \, dy \Big)}_{\left[\frac{1}{2} y^2\right]_{x^3}^1 = \frac{1}{2}(1 - x^6)} dx$$

$$= \int_0^1 \tfrac{1}{2} (x - x^7) \, dx$$

$$= \frac{1}{2} \int_0^1 (x - x^7) \, dx = \frac{1}{2} \left(\frac{1}{2} - \frac{1}{8} \right)$$

$$= \frac{1}{2} \left(\frac{4}{8} - \frac{1}{8} \right) = \frac{1}{2} \cdot \frac{3}{8} = \frac{3}{16}$$

Satz:

$$G = \{ (x,y,z) \in \mathbb{R}^3 : \phi(x,y) < z < \psi(x,y); \; (x,y) \in g \}$$

g Gebiet mit Flächeninhalt!; ϕ, ψ stetig; $f: G \to \mathbb{R}$ stetig

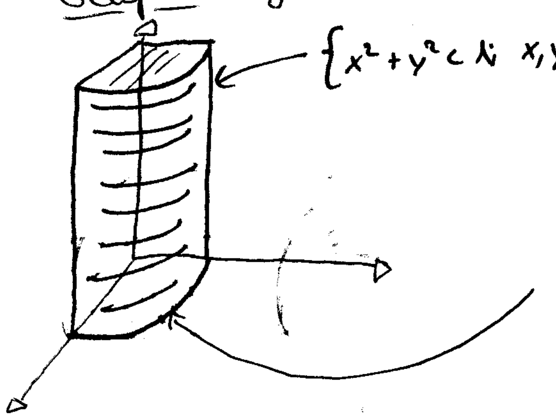
Dann gilt:

$$\int_G f(x,y,z) \, dx\, dy\, dz = \int_g \Big(\underbrace{\int_{\phi(x,y)}^{\psi(x,y)} f(x,y,z) \, dz}_{=: F(x,y)} \Big) dx\, dy$$

Beachte: Das äußere Integral kann, falls
$g = \{ (x,y) \in \mathbb{R}^2 : a(y) < x < b(y); \; y \in (c,d) \}$,
nach dem Prinzip des vorhergehenden Satzes berechnet werden.

Beispiel: $f(x,y,z) = x^2 + y^2 + z^2$

$$\{ x^2 + y^2 < 1; \; x,y,z \geq 0 \} = Z_1 \cap O_1 = \{ y = (1-x^2)^{1/2}; \; x,y,z \geq 0 \}$$



G: Schnitt aus beiden

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$$\int_G f(x,y,z)\,dx\,dy\,dz = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} x^2+y^2+z^2 \; dz\; dy\; dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} x^2 \; dz\; dy\; dx + \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} y^2+z^2 \; dz\; dy\; dx$$

$$= \int_0^1 x^2 (1-x^2)\, dx + 2 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} y^2 \; dz\, dy\, dx$$

$$= \int_0^1 x^2 - x^4 \, dx + 2 \int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \sqrt{1-x^2} \; dy\, dx$$

$$= \frac{1}{3} - \frac{1}{5} + 2 \int_0^1 \sqrt{1-x^2} \left(\int_0^{\sqrt{1-x^2}} y^2\, dy \right) dx$$

$$= \frac{2}{15} + 2 \int_0^1 \sqrt{1-x^2} \left[\frac{1}{3} (1-x^2)^{3/2} \right] dx$$

$$= \frac{2}{15} + \frac{2}{3} \int_0^1 (1-x^2)^2 \, dx$$

$$= \frac{2}{15} + \frac{2}{3} \int_0^1 (1-2x^2+x^4)\, dx$$

$$= \frac{2}{15} + \frac{2}{3} \left[1 - \frac{2}{3} + \frac{1}{5} \right]$$

$$= \frac{2}{15} + \frac{2}{3} - \frac{4}{9} + \frac{2}{15} = \frac{6 + 30 - 20 +}{45} = \frac{22}{45}$$