# 7. Übung B-Teil

**Nr. 33** Berechne mit der Halbwinkelmethode

$$\int \frac{dx}{\sqrt{1-\sin(x)}} \qquad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$

Forme $\frac{1}{\sqrt{1-\sin(x)}}$ in eine rationale Funktion um:

$$1-\sin(x) = 1-\sin\left(2\tfrac{x}{2}\right) = 1-2\sin\left(\tfrac{x}{2}\right)\cos\left(\tfrac{x}{2}\right)$$

$$= \sin^2\left(\tfrac{x}{2}\right)+\cos^2\left(\tfrac{x}{2}\right)-2\sin\left(\tfrac{x}{2}\right)\cos\left(\tfrac{x}{2}\right) = \left(\sin\left(\tfrac{x}{2}\right)-\cos\left(\tfrac{x}{2}\right)\right)^2$$

$$\Rightarrow \frac{1}{\sqrt{1-\sin(x)}} = \frac{1}{\left|\sin\left(\frac{x}{2}\right)-\cos\left(\frac{x}{2}\right)\right|} = \frac{1}{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)} \qquad \left(\text{für } -\tfrac{\pi}{2}<x<\tfrac{\pi}{2}: \cos\left(\tfrac{x}{2}\right) > \sin\left(\tfrac{x}{2}\right)\right)$$

$$= \frac{1}{\frac{1-\tan^2\left(\frac{x}{4}\right)}{1+\tan^2\left(\frac{x}{4}\right)} - \frac{2\tan\left(\frac{x}{4}\right)}{1+\tan^2\left(\frac{x}{4}\right)}} = \frac{1+\tan^2\left(\frac{x}{4}\right)}{1-2\tan\left(\frac{x}{4}\right)-\tan^2\left(\frac{x}{4}\right)}$$

Damit gilt:

$$\int \frac{1}{\sqrt{1-\sin(x)}}\,dx = \int \frac{1+\tan^2\left(\frac{x}{4}\right)}{1-2\tan\left(\frac{x}{4}\right)-\tan^2\left(\frac{x}{4}\right)}\,dx$$

$$= \int \frac{4}{1-2u-u^2}\,du \qquad \underline{\text{Subst.:}} \quad u = \tan\left(\tfrac{x}{4}\right), \quad u' = \tfrac{1}{4}\left(1+\tan^2\left(\tfrac{x}{4}\right)\right)$$

<u>Partialbruchzerlegung:</u>

$$\frac{-1}{2u+u^2-1} = \frac{A}{u+1-\sqrt{2}} + \frac{B}{u+1+\sqrt{2}}$$

$$\Rightarrow A = \frac{-1}{-1+\sqrt{2}+1+\sqrt{2}} = \frac{-1}{2\cdot\sqrt{2}}$$

$$\text{und } B = \frac{-1}{-1-\sqrt{2}+1-\sqrt{2}} = \frac{1}{2\cdot\sqrt{2}}$$

also: $4\int \frac{1}{1-2u-u^2}\,du = 4\int \frac{-1}{2\cdot\sqrt{2}}\cdot\frac{1}{u+1-\sqrt{2}} + \frac{1}{2\sqrt{2}}\cdot\frac{1}{u+1+\sqrt{2}}\,du$

$= -\frac{2}{\sqrt{2}}\int \frac{1}{u+1-\sqrt{2}}\,du + \frac{2}{\sqrt{2}}\int \frac{1}{u+1+\sqrt{2}}\,du$

$= -\sqrt{2}\cdot\left(\ln|u+1-\sqrt{2}| + c_1\right) + \sqrt{2}\left(\ln|u+1+\sqrt{2}| + c_2\right)$

$= \sqrt{2}\cdot\ln\left|\frac{u+1+\sqrt{2}}{u+1-\sqrt{2}}\right| + \underbrace{\sqrt{2}\,(c_2-c_1)}_{=:c}$

$= \sqrt{2}\cdot\ln\left(\left|\frac{\tan(\frac{x}{4})+1+\sqrt{2}}{\tan(\frac{x}{4})+1-\sqrt{2}}\right|\right) + c$

~~Nr. 34 Berechne die Integrale~~

$\cancel{a)\ \int_0^{\pi}\left|\sin(x)-\frac{2}{\pi}\right|dx = \int_0^{\arcsin(\frac{2}{\pi})} \frac{2}{\pi}-\sin(x)\,dx + \int_{\arcsin(\frac{2}{\pi})}^{\pi} \sin(x)-\frac{2}{\pi}\,dx}$

$\cancel{= \left[\frac{2}{\pi}x+\cos(x)\right]_0^{\arcsin(\frac{2}{\pi})} + \left[-\cos(x)-\frac{2}{\pi}x\right]_{\arcsin(\frac{2}{\pi})}^{\pi}}$

$\cancel{= \frac{2}{\pi}\arcsin\left(\frac{2}{\pi}\right) + \cos\left(\arcsin\left(\frac{2}{\pi}\right)\right) - 1 + 1 - 2 + \cos\left(\arcsin\left(\frac{2}{\pi}\right)\right) + \frac{2}{\pi}\arcsin\left(\frac{2}{\pi}\right)}$

$\cancel{= \frac{4}{\pi}\arcsin\left(\frac{2}{\pi}\right) + 2\cos\left(\arcsin\left(\frac{2}{\pi}\right)\right) - 2}$

$\cancel{= \frac{4}{\pi}\arcsin\left(\frac{2}{\pi}\right) + 2\sqrt{1-\left(\frac{2}{\pi}\right)^2} - 2}$

Aufgabe 34

a) $$\int_0^{\pi} \left| \sin(x) - \frac{2}{\pi} \right| dx = \int_0^{\arcsin(\frac{2}{\pi})} \frac{2}{\pi} - \sin(x)\,dx$$

$$+ \int_{\arcsin(\frac{2}{\pi})}^{\pi - \arcsin(\frac{2}{\pi})} \sin(x) - \frac{2}{\pi}\,dx + \int_{\pi - \arcsin(\frac{2}{\pi})}^{\pi} \frac{2}{\pi} - \sin(x)\,dx$$

$$= \left[ \frac{2}{\pi} x + \cos(x) \right]_0^{\arcsin(\frac{2}{\pi})} + \left[ -\cos(x) - \frac{2}{\pi} x \right]_{\arcsin(\frac{2}{\pi})}^{\pi - \arcsin(\frac{2}{\pi})}$$

$$+ \left[ \frac{2}{\pi} x + \cos(x) \right]_{\pi - \arcsin(\frac{2}{\pi})}^{\pi}$$

$$= \frac{8}{\pi} \arcsin\left(\frac{2}{\pi}\right) + 2\cos\left(\arcsin\left(\frac{2}{\pi}\right)\right)$$

$$- 4 - 2 \underbrace{\cos\left(\pi - \arcsin\left(\frac{2}{\pi}\right)\right)}_{= -\cos(\arcsin(\frac{2}{\pi}))}$$

$$= \frac{8}{\pi} \arcsin\left(\frac{2}{\pi}\right) + 4\cos\left(\arcsin\left(\frac{2}{\pi}\right)\right) - 4$$

b) $\int_{\pi/2}^{\pi} (x\cos(x)+\sin(x))\ \ln\left(\frac{\pi}{2x}\right) dx$

$= \int_{\pi/2}^{\pi} (x\sin(x))'\ \ln\left(\frac{\pi}{2x}\right) dx$

$\underset{\text{p.I.}}{=} \left[x\sin(x)\cdot \ln\left(\frac{\pi}{2x}\right)\right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} x\sin(x)\cdot \left[\ln\left(\frac{\pi}{2x}\right)\right]' dx$

$= 0 - \int_{\pi/2}^{\pi} x\cdot\sin(x)\cdot \frac{1}{\frac{\pi}{2x}}\cdot\frac{\pi}{2}\cdot\frac{(-1)}{x^2}\, dx$

$= + \int_{\pi/2}^{\pi} \sin(x)dx \quad = \left[-\cos(x)\right]_{\pi/2}^{\pi} \quad = 1$

c) $\int_0^{\pi/4} \tan^2(x) + 2\cos^2(2x)\ dx$

$= \int_0^{\pi/4} \tan^2(x)+1-1\ dx \ + 2\int_0^{\pi/4} \cos^2(2x)\ dx$

$= \int_0^{\pi/4} (\tan(x))' - 1\ dx \ + \int_0^{\pi/2} \cos^2(u)\, du$ $\qquad$ Sub.: $u = 2x$, $u' = 2$

$= \left[\tan(x) - x\right]_0^{\pi/4} + \left[\frac{1}{2}\cos(u)\sin(u)\right]_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2} 1\ du$ (nach Nr. 330)

$= 1 - \frac{\pi}{4} + \frac{1}{2}\left[x\right]_0^{\pi/2} \quad = 1$

d) $\int_0^{\pi/2} \frac{x}{1+\cos(x)}\,dx$ Subst: $u=\frac{1}{2}x$, $u'=\frac{1}{2}$

$u(0)=0$, $u(\frac{\pi}{2})=\frac{\pi}{4}$

$= 2\int_0^{\pi/4} \frac{2u}{1+\cos(2u)}\,du = 2\int_0^{\pi/4} \frac{2u}{1+\cos^2(u)-1}\,du$, denn:

$\cos(2x)=\cos^2(x)-1$

$= 2\int_0^{\pi/4} \frac{u}{\cos^2(u)}\,du = 2\int_0^{\pi/4} u \underbrace{(1+\tan^2(u))}_{(\tan(u))'}\,du$

$\overset{\text{pI}}{=} 2\Big[u\cdot\tan(u)\Big]_0^{\pi/4} - 2\int_0^{\pi/4} \tan(u)\,du$

$= 2\Big[u\tan(u) + \ln(\cos(u))\Big]_0^{\pi/4}$

$= 2\left(\frac{\pi}{4} - \ln\left(\frac{\sqrt{2}}{2}\right)\right) = \frac{\pi}{2} - \ln(2)$

## Nr. 35 Berechne die Integrale

a) $\int \sin(\sqrt{x})\,dx = \int \frac{\sqrt{x}}{\sqrt{x}}\sin(\sqrt{x})\,dx$ Subst.: $y=\sqrt{x}$, $y'=\frac{1}{2\sqrt{x}}$

$= 2\int y\sin(y)\,dy = 2\int y\cdot(-\cos(y))'\,dy$

$\overset{\text{pI}}{=} 2\left[-y\cos(y) + C_1 + \int \cos(y)\,dy\right]$

$= 2\left(-y\cos(y) + \sin(y) + C\right)$

$= 2\sin(\sqrt{x}) - 2\sqrt{x}\cos(\sqrt{x}) + C$

b) $\int \frac{x^3}{x^8+1}\,dx$ Subst.: $y = x^4$, $y' = 4x^3$

$= \frac{1}{4}\int \frac{1}{1+y^2}\,dy = \frac{1}{4}\arctan(y) + C$

$= \frac{1}{4}\arctan(x^4) + C$

c) $\int \frac{1}{\cos(x)}\,dx = \int \frac{1+\tan^2(\frac{x}{2})}{1-\tan^2(\frac{x}{2})}\,dx$ Subst.: $u = \tan(\frac{x}{2})$, $u' = \frac{1}{2}(1+\tan^2(\frac{x}{2}))$

$= 2\int \frac{1}{1-u^2}\,du$

$= 2\cdot \operatorname{artanh}(u) + C$ , denn : $0 < u = \tan(\frac{x}{2}) < 1$ für $0 < x < \frac{\pi}{2}$

$= 2\cdot \operatorname{artanh}(\tan(\frac{x}{2})) + C$

e) $\int \frac{x^5}{x^4+1}\,dx = \int \frac{(x^4+1)\cdot x - x}{x^4+1}\,dx$

$= \int x\,dx - \int \frac{x}{x^4+1}\,dx$ Subst.: $u = x^2$, $u' = 2x$

$= \frac{1}{2}x^2 + c_1 - \frac{1}{2}\int \frac{1}{1+u^2}\,du$

$= \frac{1}{2}x^2 + c_1 - \frac{1}{2}\arctan(u) + c_2$

$= \frac{1}{2}x^2 - \frac{1}{2}\arctan(x^2) + C$

Nr. 36 Berechne mittels Partialbruchzerlegung

$$\int \frac{x+3}{(x-1)(x^2+1)^2}\,dx \quad \text{in } \mathbb{R}\setminus\{1\}.$$

Partialbruchzerlegung:

$$\frac{x+3}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$\begin{aligned} x+3 &= A(x^4+2x^2+1) + (Bx+C)(x^3-x^2+x-1) + (Dx+E)(x-1) \\ &= x^4(A+B) + x^3(-B+C) + x^2(2A+B-C+D) \\ &\quad + x(-B+C-D+E) + (A-C-E) \end{aligned}$$

Koeffizientenvergleich:

i) $A+B=0 \iff A=-B$

ii) $-B+C=0 \iff C=B$

iii) $2A+B-C+D=0 \iff -2B+B-B+D=0$
$\iff D=2B$

iv) $-B+C-D+E=1 \iff -2B+E=1 \iff E=1+2B$

v) $A-C-E=3 \iff -B-B-1-2B=3 \iff B=-1$

Also: $A=1,\ B=-1,\ C=-1,\ D=-2,\ E=-1$,

d.h. $\frac{x+3}{(x-1)(x^2+1)^2} = \frac{1}{x-1} + \frac{-x-1}{x^2+1} + \frac{-2x-1}{(x^2+1)^2}$

$$\Rightarrow \int \frac{x+3}{(x-1)(x^2+1)^2}\,dx = \underbrace{\int \frac{1}{x-1}\,dx}_{I_1} + \underbrace{\int \frac{-x-1}{x^2+1}\,dx}_{I_2} + \underbrace{\int \frac{-2x-1}{(x^2+1)^2}\,dx}_{I_3}$$

$$I_1 = \int \frac{1}{x-1}\, dx = \ln|x-1| + c_1$$

$$I_2 = -\int \frac{x}{x^2+1}\, dx - \int \frac{1}{x^2+1}\, dx$$

$$\underset{\substack{\uparrow \\ y = x^2+1 \\ y' = 2x}}{=} -\frac{1}{2}\int \frac{1}{y}\, dy - \arctan(x) + c_2'$$

$$= -\frac{1}{2} \ln|y| + c_2'' - \arctan(x) + c_2'$$

$$= -\frac{1}{2} \ln(x^2+1) - \arctan(x) + C_2$$

$$I_3 = -\int \frac{2x}{(x^2+1)^2}\, dx - \int \frac{1}{(x^2+1)^2}\, dx$$

$$\underset{\substack{\uparrow \\ y = x^2+1 \\ y' = 2x}}{=} -\int \frac{1}{y^2}\, dy - \int \left(\frac{1}{1+x^2} - \frac{x^2}{(1+x^2)^2}\right) dx$$

$$= \frac{1}{y} + c_3' - \arctan(x) + c_3'' + \int \frac{x^2}{(1+x^2)^2}\, dx$$

$$= \frac{1}{x^2+1} + c_3' - \arctan(x) + c_3'' + \int x \left(\frac{(-1)}{2} \frac{1}{1+x^2}\right)' dx$$

$$= \frac{1}{x^2+1} - \arctan(x) + c_3' + c_3'' + \left(-\frac{1}{2} \frac{x}{1+x^2} \underset{+c_3'''}{\wedge} + \frac{1}{2} \int \frac{1}{1+x^2}\, dx\right)$$

$$= \frac{1}{x^2+1} - \arctan(x) - \frac{1}{2} \frac{x}{1+x^2} + \frac{1}{2} \arctan x + C_3$$

$$= \frac{1 - \frac{x}{2}}{x^2+1} - \frac{1}{2} \arctan(x) + C_3$$

Also insgesamt:

$$\int \frac{x+3}{(x-1)(x^2+1)}\, dx = \ln|x-1| - \frac{1}{2} \ln(x^2+1) - \arctan(x) + \frac{1 - \frac{x}{2}}{x^2+1} - \frac{1}{2} \arctan(x) + C$$

$$= \ln\left(\frac{|x-1|}{\sqrt{x^2+1}}\right) - \frac{3}{2}\arctan(x) + \frac{2-x}{2(x^2+1)} + C$$

## Nr. 37 Berechne die Integrale

a) $$\int (x+2)\sqrt{x^2+1}\,dx = \underbrace{\int x\sqrt{x^2+1}\,dx}_{I_1} + 2\cdot\underbrace{\int\sqrt{x^2+1}\,dx}_{I_2}$$

$$I_1 \underset{\substack{u=x^2+1\\u'=2x}}{=} \frac{1}{2}\int\sqrt{u}\,du = \frac{1}{3}(\sqrt{u})^3 + C = \frac{1}{3}\sqrt{x^2+1}^3$$

$$I_2 \underset{\substack{x=\sinh(y)\\dx=\cosh(y)dy}}{=} \int\sqrt{\sinh^2(y)+1}\,\cosh(y)dy = \int\cosh^2(y)\,dy$$

$$\int\cosh^2(y)dy \underset{PI}{=} \sinh(y)\cosh(y) - \int\underbrace{\cosh^2(y)-1}_{\sinh^2(y)}\,dy$$

$$\iff \int\cosh^2(y)dy = \frac{1}{2}(\sinh(y)\cosh(y) + y) + C$$

$$\Rightarrow I_2 \underset{y=\operatorname{arsinh}(x)}{=} \frac{1}{2}(x\cdot\cosh(\operatorname{arsinh}(x)) + \operatorname{arsinh}(x)) + C$$

$$= \frac{1}{2}(x\cdot\cosh(\operatorname{sgn}(x)\operatorname{arcosh}(\sqrt{x^2+1})) + \operatorname{arsinh}(x)) + C$$

$$= \frac{1}{2}(x\sqrt{x^2+1} + \operatorname{arsinh}(x)) + C$$

$$\Rightarrow \int (x+2)\sqrt{x^2+1}\,dx = \frac{1}{3}\sqrt{x^2+1}^3 + x\sqrt{x^2+1} + \operatorname{arsinh}(x) + C$$

c) $$\int \frac{(x+2)^2}{\sqrt{-x^2-4x+3}}\,dx = \int \frac{(x+2)^2}{\sqrt{1-(x+2)^2}}\,dx$$ <u>Subst.</u>: $t = x+2$

$$= \int \frac{t^2}{\sqrt{1-t^2}}\,dt = \frac{1}{2}\int \frac{2t^2+1-1}{\sqrt{1-t^2}}\,dt$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-t^2}}\,dt - \frac{1}{2}\int \frac{1-2t^2}{\sqrt{1-t^2}}\,dt$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-t^2}}\,dt - \frac{1}{2}\int \underbrace{\sqrt{1-t^2} + t\frac{t}{\sqrt{1-t^2}}}_{(t\sqrt{1-t^2})'}\,dt$$

$$= \frac{1}{2}\arcsin(t) - \frac{1}{2}t\sqrt{1-t^2} + C$$

$$= \frac{1}{2}\arcsin(x+2) - \frac{1}{2}(x+2)\sqrt{1-(x+2)^2} + C.$$

b) $$\int \frac{x+\frac{1}{2}}{\sqrt{x^2-x+\frac{5}{4}}}\,dx = \int \frac{x+\frac{1}{2}}{\sqrt{1+(x-\frac{1}{2})^2}}\,dx$$ <u>Subst.</u>: $t = x-\frac{1}{2}$, $x+\frac{1}{2} = t+1$

$$= \int \frac{t-1}{\sqrt{1+t^2}}\,dt = \int \underbrace{\frac{t}{\sqrt{1+t^2}}}_{=(\sqrt{1+t^2})'}\,dt + \int \frac{1}{\sqrt{1+t^2}}\,dt$$

$$= \sqrt{1+t^2} + \operatorname{arsinh}(t) + C$$

$$= \sqrt{1+(x-\tfrac{1}{2})^2} + \operatorname{arsinh}(x-\tfrac{1}{2}) + C$$