6. Übung B-Teil

Nr. 28 Berechne die Ableitungen und zeige, dass die Funktionen glm. stetig sind.

a) $f(x) = \int_1^{\ln(x)} \sqrt{e^t+1} \quad , x \in (1,\infty)$

Dann gilt für $f'(x)$ nach 1. Hauptsatz:

$$f'(x) = \sqrt{e^{\ln(x)}+1} \cdot (\ln(x))'$$

$$= \sqrt{x+1} \cdot \frac{1}{x}$$

Es gilt: $f'(x) < \sqrt{2} \quad \forall x \in (1,\infty)$

denn: $f'(x) = \frac{\sqrt{x+1}}{x} \underset{x\in(1,\infty)}{<} \frac{\sqrt{x+x}}{x} = \frac{\sqrt{2}}{\sqrt{x}} \underset{x\in(1,\infty)}{<} \sqrt{2}$

Dann folgt die glm. Stetigkeit von f aus A-Teil-Aufgabe Nr. 20.

b) $g(x) = \int_{\ln(\ln(x))}^{\ln(x)} \sqrt{e^t+1} = \int_{\ln(\ln(x))}^{a} \sqrt{e^t+1}\,dt + \int_a^{\ln(x)} \sqrt{e^t+1}\,dt$ für

$a \in \mathbb{R}$ mit $\ln(\ln(x)) \leq a \leq \ln(x)$

$$= \int_a^{\ln(x)} \sqrt{e^t+1}\,dt - \int_a^{\ln(\ln(x))} \sqrt{e^t+1}\,dt$$

1. Hauptsatz liefert:

$$g'(x) = \sqrt{e^{\ln(x)}+1} \cdot (\ln(x))' - \sqrt{e^{\ln(\ln(x))}+1} \cdot (\ln(\ln(x)))'$$

$$= \frac{\sqrt{x+1}}{x} - \frac{\sqrt{\ln(x)+1}}{\underbrace{\ln(x)\cdot x}_{\substack{\geq 0,\text{ da} \\ x\in(2,\infty)}}} \leq \frac{\sqrt{x+1}}{x} < \sqrt{2}$$

Dann folgt die glm. Stetigkeit wieder mit A20

Nachtrag zu Nr. 28 b)

für die glm. Stetigkeit ist zu zeigen:

$$|g'(x)| \leq M \quad \forall x \in D(g) = (2, \infty).$$

- für $x \in [e, \infty)$:

$$\frac{\sqrt{\ln(x)+1}}{\ln(x) \cdot x} \leq \frac{\sqrt{\ln(x)+1}}{x} \leq \frac{\sqrt{x+1}}{x}$$

$$\Leftrightarrow \quad g'(x) = \frac{\sqrt{x+1}}{x} - \frac{\sqrt{\ln(x)+1}}{\ln(x) \cdot x} \geq 0$$

$$\Rightarrow \quad |g'(x)| \leq \sqrt{2} \quad \forall x \in [e, \infty).$$

- für $x \in (2, e)$:

auf $(2, e)$ ist g' monoton wachsend

$$\Rightarrow \quad g'(2) < g'(x) < g'(e)$$

$$\Leftrightarrow \quad \frac{\sqrt{3} \cdot \ln(2) - \sqrt{\ln 2 + 1}}{2 \cdot \ln(2)} < g'(x) < \frac{\sqrt{e+1} - \sqrt{2}}{e}$$

$$\Leftrightarrow \quad -0{,}07 < g'(x) < 0{,}189$$

$$\Rightarrow \quad |g'(x)| < 0{,}189 \quad \forall x \in (2, e)$$

$$\Rightarrow \quad |g'(x)| < \sqrt{2} \quad \forall x \in (2, \infty)$$

und damit ist g glm. stetig.

Nr. 29 Berechne die Integrale.

a) $\int_0^{\pi} x^3 \sin(x)\,dx = \int_0^{\pi} x^3 \cdot (-\cos(x))'\,dx$

$\underset{PI}{=} -x^3\cos(x)\Big|_0^{\pi} + 3\int_0^{\pi} x^2\cos(x)\,dx = \pi^3 + 3\int_0^{\pi} x^2 (\sin(x))'\,dx$

$\underset{PI}{=} \pi^3 + 3\left(x^2\sin(x)\Big|_0^{\pi} - 2\int_0^{\pi} x\sin(x)\,dx\right)$

$= \pi^3 - 6\int_0^{\pi} x\,(-\cos(x))'\,dx = \pi^3 + 6x\cos(x)\Big|_0^{\pi} - 6\int_0^{\pi}\cos(x)\,dx$

$= \pi^3 - 6\pi - 6(\sin(x))\Big|_0^{\pi} = \pi^3 - 6\pi$

b) $\int_0^{1/2} x\cdot\arcsin(x^2)\,dx = \int_0^{1/2} (\tfrac{1}{2}x^2)'\arcsin(x^2)\,dx$

$\underset{PI}{=} \frac{1}{2}x^2\arcsin(x^2)\Big|_0^{1/2} - \frac{1}{2}\int_0^{1/2} x^2\cdot(\arcsin(x^2))'\,dx$

$= \frac{1}{2}x^2\arcsin(x^2)\Big|_0^{1/2} - \frac{1}{2}\int_0^{1/2} x^2\frac{2x}{\sqrt{1-x^4}}\,dx$

$= \frac{1}{2}x^2\arcsin(x^2)\Big|_0^{1/2} - \int_0^{1/2}\frac{x^3}{\sqrt{1-x^4}}\,dx$

Substitution: $u = x^4,\ u' = 4x^3$

$= \frac{1}{2}x^2\arcsin(x^2)\Big|_0^{1/2} - \frac{1}{4}\int_0^{1/16}\underbrace{\frac{1}{\sqrt{1-u}}}_{=2(-\sqrt{1-u})'}\,du$

$= \frac{1}{2}x^2\arcsin(x^2)\Big|_0^{1/2} + \frac{1}{2}\sqrt{1-u}\Big|_0^{1/16}$

$= \frac{1}{8}\arcsin(\frac{1}{4}) + \frac{\sqrt{15}}{8} - \frac{1}{2}$

c) $\int_0^{1/2} \arcsin(x)\,dx = \int_0^{1/2} 1 \cdot \arcsin(x)\,dx$

$= \int_0^{1/2} (x)' \cdot \arcsin(x)\,dx$

$\underset{p.I}{=} \left[x \cdot \arcsin(x)\right]_0^{1/2} - \int_0^{1/2} x \cdot (\arcsin(x))'\,dx$

$= \frac{1}{2} \cdot \arcsin\left(\frac{1}{2}\right) - \int_0^{1/2} x \cdot \frac{1}{\sqrt{1-x^2}}\,dx$

$= \frac{1}{2} \cdot \arcsin\left(\frac{1}{2}\right) + \int_0^{1/2} \left(\sqrt{1-x^2}\right)'\,dx$

$= \frac{1}{2} \arcsin\left(\frac{1}{2}\right) + \left(\sqrt{1-x^2}\right)_0^{1/2} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$

Nr. 31 Berechne die Integrale

a) $\int_e^{e^2} \frac{\ln(\ln(x))}{x}\,dx$ Subst. $y = \ln(x)$, $y' = \frac{1}{x}$

$= \int_1^2 \ln(y)\,dy = \int_1^2 1 \cdot \ln(y)\,dy$

$\underset{pI}{=} y\ln(y)\Big|_1^2 - \int_1^2 1\,dy = \left[y\ln(y) - y\right]_1^2$

$= 2\ln(2) - 2 + 1$

$= 2\ln(2) - 1$

b) $\int_0^{\pi} \cos(x)\sin(\frac{x}{2})\,dx = \int_0^{\pi} 2\cos^2(\frac{x}{2})\sin(\frac{x}{2}) - \sin(\frac{x}{2})\,dx$

(denn: $\cos(2x) = 2\cos^2(x) - 1$)

Subst. $u = \cos(\frac{x}{2})$, $u' = -\sin(\frac{x}{2}) \cdot \frac{1}{2}$

bzw. $w = \frac{x}{2}$, $w' = \frac{1}{2}$

$$= -4\int_1^0 u^2\,du - 2\int_0^{\pi/2} \sin(w)\,dw$$

$$= 4\int_0^1 u^2\,du + 2\left[\cos(w)\right]_0^{\pi/2}$$

$$= \frac{4}{3}u^3\Big|_0^1 + 2\cos(\tfrac{\pi}{2}) - 2\cos(0) = \frac{4}{3} - 2 = -\frac{2}{3}$$

c) $\int_0^1 \frac{1-x}{1+\sqrt{x(2-x)}}\,dx$

Substituiere: $y = 1 + \sqrt{x(2-x)}$

$y' = \frac{1-x}{\sqrt{x(2-x)}}$

$y(0) = 1$, $y(1) = 2$

$$= \int_0^1 \frac{1-x}{1+\sqrt{x(2-x)}} \cdot \frac{\sqrt{x(2-x)}}{\sqrt{x(2-x)}}\,dx$$

$$= \int_1^2 \frac{y-1}{y}\,dy = \int_1^2 1 - \frac{1}{y}\,dy = \left[y - \ln y\right]_1^2$$

$$= 2 - \ln(2) - 1 = 1 - \ln(2)$$

Nr. 30 Zeige für $m, n \in \mathbb{N}$, $n \geq 2$, die Formeln

a) $\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$

b) $\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$

c) $\int \sin^m(x)\cos^n(x)\,dx = \frac{\sin^{m+1}(x)\cos^{n-1}(x)}{m+n} + \frac{n-1}{m+n}\int \sin^m(x)\cdot\cos^{n-2}(x)\,dx$

dazu:

a) $\int \cos^n(x)\,dx = \int \cos^{n-1}(x)\cdot\cos(x)\,dx = \int \cos^{n-1}(x)\cdot(\sin(x))'\,dx$

$\underset{\text{P.I}}{=} \sin(x)\cdot\cos^{n-1}(x) - \int \sin(x)\cdot[\cos^{n-1}(x)]'\,dx$

$= \sin(x)\cdot\cos^{n-1}(x) - \int \sin(x)\cdot(n-1)\cos^{n-2}(x)\cdot(-\sin(x))\,dx$

$= \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)\cdot\sin^2(x)\,dx$

$= \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)\left(\sin^2(x) + \cos^2(x) - \cos^2(x)\right)dx$

$= \sin(x)\cdot\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)\,(1 - \cos^2(x))\,dx$

$\Longleftrightarrow [(n-1)+1]\int \cos^n(x)\,dx = \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\,dx$

$\Longleftrightarrow \int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$

b) $\int \sin^n(x)\,dx = \int \sin^{n-1}(x)\cdot\sin(x)\,dx = \int \sin^{n-1}(x)\cdot(-\cos(x))'\,dx$

$= -\sin^{n-1}(x)\cos(x) + \int \cos(x)\cdot[\sin^{n-1}(x)]'\,dx$

$= -\sin^{n-1}(x)\cos(x) + \int \cos(x)\cdot(n-1)\sin^{n-2}(x)\cdot\cos(x)\,dx$

$= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx - (n-1)\int \sin^n(x)\,dx$

$$\Leftrightarrow \quad n\int \sin^n(x)\,dx = -\sin^{n-1}(x)\cdot\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx$$

$$\Leftrightarrow \quad \int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

c) $$\int \sin^m(x)\cos^n(x)\,dx = \int \sin^m(x)\cos^{n-1}(x)\underbrace{\cos(x)}_{=(\sin(x))'}\,dx$$

$$\overset{pI}{=} \sin(x)\cdot\sin^m(x)\cdot\cos^{n-1}(x) - \int \sin(x)\left(\sin^m(x)\cdot\cos^{n-1}(x)\right)'\,dx$$

$$= \sin^{m+1}(x)\cos^{n-1}(x) - \int \sin(x)\Big[m\sin^{m-1}(x)\cdot\cos^n(x)$$
$$(n-1)\cos^{n-2}(x)\cdot\sin^{m+1}(x)\Big]\,dx$$

$$= \sin^{m+1}(x)\cos^{n-1}(x) - m\int \sin^m(x)\cos^n(x)\,dx + (n-1)\int \sin^{m+2}(x)\cos^{n-2}(x)\,dx$$

$$\underset{\substack{\uparrow\\ \sin^{m+2}(x)\\ =\sin^m(x)(1-\cos^2(x))}}{=} \sin^{m+1}(x)\cos^{n-1}(x) - m\int \sin^m(x)\cos^n(x)\,dx$$
$$+ (n-1)\int \sin^m(x)\cos^{n-2}\,dx$$
$$- (n-1)\int \sin^m(x)\cos^n\,dx$$

$$\Leftrightarrow (1+m+n-1)\int \sin^m(x)\cos^n(x)\,dx = \sin^{m+1}(x)\cos^{n-1}(x) + (n-1)\int \sin^m(x)\cos^{n-2}(x)\,dx$$

$$\Leftrightarrow \int \sin^m(x)\cos^n(x)\,dx = \frac{\sin^{m+1}(x)\cos^{n-1}(x)}{m+n} + \frac{n-1}{m+n}\int \sin^m(x)\cos^{n-2}(x)\,dx$$

$\square$

Nr. 32 Berechne mit der Halbwinkelmethode

$$\int \frac{1}{-2+2\sin(x)}\,dx \quad , x \in (0, \tfrac{\pi}{2})$$

dazu: es gilt: $\sin(x) = \dfrac{2\tan(\frac{x}{2})}{1+\tan^2(\frac{x}{2})}$.

Damit:

$$\int \frac{1}{-2+2\sin(x)}\,dx = \int \frac{1}{-2+2\cdot\frac{2\tan(\frac{x}{2})}{1+\tan^2(\frac{x}{2})}}\,dx$$

Subst.: $y = \tan(\frac{x}{2})$

$y' = \frac{1}{2}(1+\tan^2(\frac{x}{2}))$

$$= \int \frac{1+\tan^2(\frac{x}{2})}{4\tan(\frac{x}{2})-2-2\tan(\frac{x}{2})}\,dx$$

$$= 2\int \frac{1}{4y-2-2y^2}\,dy = \int \frac{-1}{y^2-2y+1}\,dy$$

$$= \int \frac{-1}{(y-1)^2}\,dy = \frac{1}{y-1} + C$$

$$= \frac{1}{\tan(\frac{x}{2})-1} + C.$$