Aufgabe A 57

(a) $$\int_0^{\pi/2} \frac{\sin^3 x \cos x}{1+2\cos x}\, dx = \int_0^{\pi/2} \frac{(1-\cos^2 x)\cos x}{1+2\cos x} \sin x \, dx$$

$$\underset{\frac{dv}{dx}=-\sin x}{\overset{v=\cos x}{=}} -\int_1^0 \frac{(1-v^2)v}{1+2v}\, dv = -\int_0^1 \frac{v^3-v}{2v+1}\, dv$$

Polynomdivision

$$(v^3 - v) : (2v+1) = \frac{1}{2}v^2 - \frac{1}{4}v - \frac{3}{8} + \frac{3}{8}\,\frac{1}{2v+1}$$

$$\begin{array}{l}
-(v^3 + \frac{1}{2}v^2) \\
\hline
\quad -\frac{1}{2}v^2 - v \\
\quad -(-\frac{1}{2}v^2 - \frac{1}{4}v) \\
\hline
\qquad -\frac{3}{4}v \\
\qquad -(-\frac{3}{4}v - \frac{3}{8}) \\
\hline
\qquad\quad \frac{3}{8}
\end{array}$$

$$= -\left[\frac{1}{6}v^3 - \frac{1}{8}v^2 - \frac{3}{8}v + \frac{3}{16}\log(2v+1)\right]_0^1$$

$$= -\left(\frac{1}{6} - \frac{1}{8} - \frac{3}{8} + \frac{3}{16}\log 3\right)$$

$$= \frac{1}{3} - \frac{3}{16}\log 3$$

(b) $\int_0^1 \sqrt{1+x^2}\,dx$

Wir substituieren $x = \sinh v \Rightarrow \frac{dx}{dv} = \cosh v$.
Umgekehrt ist nach Definition $2x = e^v - e^{-v}$
$\Leftrightarrow (e^v)^2 - 2xe^v - 1 = 0$, und die pq-Formel
liefert $e^v = x \pm \sqrt{x^2+1}$, also $v = \log(x + \sqrt{x^2+1})$.

Warum?

$$\Rightarrow \int_0^1 \sqrt{1+x^2}\,dx = \int_0^{\log(\sqrt{2}+1)} \cosh^2 v\,dv$$

$$= \frac{1}{4}\int_0^{\log(\sqrt{2}+1)} \left(e^{2x} + 2 + e^{-2x}\right) dx$$

$$= \frac{1}{4}\left[\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}\right]_0^{\log(\sqrt{2}+1)}$$

$$= \frac{1}{4}\left(\frac{1}{2}(\sqrt{2}+1)^2 + 2\log(\sqrt{2}+1) - \frac{1}{2}\left(\frac{1}{\sqrt{2}+1}\right)^2\right)$$

$$= \frac{1}{8}\left(3 + 2\sqrt{2} + 4\log(\sqrt{2}+1) - \left(\frac{\sqrt{2}-1}{1}\right)^2\right)$$

$$= \frac{1}{8}\left(4\sqrt{2} + 4\log(\sqrt{2}+1)\right)$$

$$= \frac{1}{2}\sqrt{2} + \frac{1}{2}\log(\sqrt{2}+1)$$

(c) $$\int_0^{\pi/2} \frac{dx}{3+4\tan x} = \lim_{a \nearrow \frac{\pi}{2}} \int_0^a \frac{dx}{3+4\tan x} = \lim_{a \nearrow \frac{\pi}{2}} \int_0^{\tan a} \frac{dv}{(3+4v)(1+v^2)}$$

$$v = \tan x$$
$$\frac{dv}{dx} = \frac{1}{\cos^2 x} = 1+v^2$$

<u>Partialbruchzerlegung</u>

$$\frac{1}{(3+4v)(1+v^2)} = \frac{A}{3+4v} + \frac{Bv+C}{1+v^2}$$

$$\rightsquigarrow A = \frac{16}{25}, \quad B = -\frac{4}{25}, \quad C = \frac{3}{25}$$

$$\Rightarrow \int_0^{\pi/2} \frac{dx}{3+4\tan x} = \lim_{b \nearrow \infty} \left\{ \frac{16}{25} \int_0^b \frac{dv}{3+4v} + \frac{1}{25} \int_0^b \frac{-4v+3}{v^2+1}\, dv \right\}$$

$$\longrightarrow \int_0^b \frac{dv}{3+4v} = \left[\frac{1}{4} \log|3+4v| \right]_0^b$$

$$\longrightarrow \int_0^b \frac{-4v+3}{v^2+1}\, dv = -2 \int_0^b \frac{2v}{v^2+1}\, dv + 3 \int_0^b \frac{dv}{1+v^2}$$

$$= -2 \left[\log|v^2+1| \right]_0^b + 3 \left[\arctan v \right]_0^b$$

$$\Rightarrow \int_0^{\pi/2} \frac{dx}{3+4\tan x} = \lim_{b \nearrow \infty} \left\{ \frac{4}{25} \log\left(\frac{|3+4b|}{3} \right)^{2} - \frac{2}{25} \log|b^2+1| + \frac{3}{25} \arctan b \right\}$$

$$= \lim_{b \nearrow \infty} \left\{ \frac{2}{25} \log \underbrace{\frac{(3+4b)^2}{9(b^2+1)}}_{= \frac{(\frac{3}{b}+4)^2}{9(1+\frac{1}{b^2})}} + \frac{3}{25} \arctan b \right\}$$

$$= \frac{2}{25} \log \frac{16}{9} + \frac{3}{25} \cdot \frac{\pi}{2}$$

$$= \frac{8}{25} \log 2 - \frac{4}{25} \log 3 + \frac{3}{50} \pi$$

(d) $$\int_0^a x^2\sqrt{a^2-x^2}\,dx \underset{\substack{x=a\cdot\sin(t)\\ \frac{dx}{dt}=a\cdot\cos(t)}}{=} \int_0^{\pi/2} a^2\sin^2(t)\underbrace{\sqrt{a^2-a^2\sin^2(t)}}_{a\cdot\cos(t)\,dt}$$

$$= a^3|a|\int_0^{\pi/2}\sin^2(t)\cdot\cos^2(t)\,dt$$

$$= \mathrm{sign}(a)\,|a|^4\int_0^{\pi/2}\left[\tfrac{1}{2}\sin(2t)\right]^2 dt$$

$$\underset{\substack{y=2t\\ \frac{dy}{dt}=2}}{=} \frac{\mathrm{sign}(a)\,|a|^4}{4}\int_0^{\pi}\sin(y)^2\,\frac{dy}{2}$$

$$= \frac{\mathrm{sign}(a)\,|a|^4}{8}\left[\frac{1}{2}\left(y-\sin(y)\cos(y)\right)\right]_0^{\pi}$$

$$= \mathrm{sign}(a)\,|a|^4\,\frac{\pi}{16}.$$

NR:

$$\int\sin^2(y)\,dy = \int\left[1-\cos^2(y)\right]dy$$

$$\underset{P.I.}{=} y-\left[\sin(y)\cos(y)+\int\sin^2(y)\,dy\right]$$

$$\Rightarrow \int\sin^2(y)\,dy = \frac{1}{2}\left(y-\sin(y)\cos(y)\right)$$