HöMa 11 II

$$u'' - u' \cos t + u \sin t = \sin t$$

A) homogene DGL

i) $u_1(t) = e^{\sin t}$

$\Rightarrow u_1' = \sin t \cdot e^{\sin t} \quad \Rightarrow u_1'' = (\cos^2 t - \sin t) \cdot e^{\sin t}$

$\Rightarrow (\cos^2 t - \sin t)\cancel{e^{\sin t}} - \cos^2 t \cdot \cancel{e^{\sin t}} + \sin t \, \cancel{e^{\sin t}} \overset{!}{=} 0$

$\Rightarrow 0 \overset{!}{=} 0$

$\Rightarrow u_1$ ist Lsg der homogenen DGL

ii) Satz 9.3 oder Prinzip von d'Umbert

$u_2 = c(x)\, u_1$

$u_2' = u_1' c + u_1 c' \qquad u_2'' = u_1 c'' + 2u_1' c' + u_1'' c$

-> einsetzen in DGL

$\Rightarrow e^{\sin t}\left(c'' + \cancel{2}c' \cos t + c\cancel{\cos^2 t} - \cancel{c \sin t} - \cancel{c' \cos t} - \cancel{c\cos^2 t} + \cancel{c \sin t}\right) = 0$

$\Rightarrow c'' + c' \cos t = 0$

Separation

$\Rightarrow \frac{c''}{c'} = -\cos t \qquad \Rightarrow \int \frac{dc'}{c'} = \int -\cos t \, dt$

$\Rightarrow \ln c' = -\sin t + D^*$

$\Rightarrow c' = e^{-\sin t} \cdot D$

$c = \int e^{-\sin t} dt \cdot D$

$\Rightarrow u_2 = c\, u_1$
$$= e^{\sin t} \int e^{-\sin t}\, dt$$

B) inhomogene DGL $u'' - u'\cos t + u \sin t = \sin t$

Satz 9.6

Wronski-Determinante

$$\left[\, W(x) = \det \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} = u_1 u_2' - u_2 u_1' \right.$$

$$W(x) = e^{\sin x} \left(\cos x\, e^{\sin x} \int e^{-\sin x}\, dx + e^{\sin x} \cdot e^{-\sin x} \right)$$

$$- \cos x\, e^{\sin x}\, e^{\sin x} \int e^{-\sin x}\, dx$$

$$= e^{\sin x}\, e^{\sin x}\, e^{-\sin x} = e^{\sin x} \neq 0 \quad \forall x \in \mathbb{R}$$

$$u_0(x) = \sum_i u_i(x)\, y_i(x) \qquad y_i = \int \frac{\det W_i(x)}{\det W(x)}\, dx$$

W_i ist eine Wronski-Matrix, deren i-te Spalte durch $\begin{pmatrix} 0 \\ 1 \\ \vdots \\ f \end{pmatrix}$ ersetzt ~~werden~~ worden ist. (f ist die Inhomogenität)

$$\det W_1(t) = \det \begin{pmatrix} 0 & u_2 \\ f & u_2' \end{pmatrix} = -f u_2$$

$$\det W_2(t) = \det \begin{pmatrix} u_1 & 0 \\ u_1' & f \end{pmatrix} = f \cdot u_1$$

$$u_0(t) = u_1 \cdot \int \frac{-f u_2}{\det W}\, dt + u_2 \cdot \int \frac{f u_1}{\det W}\, dt$$

$$= e^{\sin t}\int \frac{-\sin t\, e^{+\sin t}\int -\sin t\, dt}{e^{\sin t}}\, dt + e^{\sin t}\int e^{-\sin t}\, dt \cdot \int \frac{\sin t\, e^{\sin t}}{\cancel{e^{\sin t}}}\, dt$$

$$= e^{\sin t}\int -\sin t\left(\int e^{-\sin t}\, dt\right) dt - \cos t\, e^{\sin t}\int e^{-\sin t}\, dt$$

ii) Satz 9.5

$$u(t) = c_1 u_1(t) + c_2 u_2(t) + u_0(t) \qquad c_1, c_2 \in \mathbb{R}$$

BS1

$$u' = \frac{u}{x} + \frac{\sqrt{x^2-u^2}}{x} \qquad |u| \leq |x|$$

$$= \frac{u}{x} + \frac{|x| \cdot \sqrt{1-\left(\frac{u}{x}\right)^2}}{x}$$

$$= \frac{u}{x} + \operatorname{sign} x \sqrt{1-\left(\frac{u}{x}\right)^2} \qquad y := \frac{u}{x}$$

$$u' := f(y) = y + \lambda\sqrt{1-y^2} \qquad \lambda \in \{\pm 1\}$$

$$y' = \frac{u'}{x} - \frac{u}{x^2} = \frac{1}{x} f(y) - \frac{y}{x}$$

Sonderfall: $f(y) = y \Rightarrow y^2 = 1 \Rightarrow y = \pm 1 \Rightarrow u(x) = \pm x$

Separation

$$\int \frac{1}{x}\, dx = \int \frac{dx}{f(y)-y} = \lambda\int \frac{dy}{\sqrt{1-y^2}} = \lambda \arcsin y - c$$

$$\Rightarrow \lambda \cdot \arcsin y = \ln|x| + c$$

$\Rightarrow \arcsin \lambda y = \ln|x| + c$ da λ Vorzeichen + arcsin ungerade

$\Rightarrow y = \lambda \cdot \sin(\ln|x| + c)$

Probe:

$$u' = \lambda \cdot \sin(\ln|x|+c) + \lambda \cancel{x} \cos(\ln|x|+c) \cdot \frac{1}{\cancel{x}}$$

$$\Rightarrow x u' - u - \sqrt{x^2 - u^2}$$

$$= \cancel{x \lambda \sin(\ln|x|+c)} + x \lambda \cos(\ln|x|+c) - \cancel{\lambda x \sin(\ln|x|+c)}$$

$$- \sqrt{x^2 - \underbrace{\lambda^2}_{1} x^2 \sin^2(\ln|x|+c)} \overset{!}{=} 0$$

$$\Rightarrow x \lambda \cos(\ln|x|+c) = |x| \sqrt{1 - \sin^2(\ln|x|+c)}$$

$$= |x| \cdot |\cos(\ln|x|+c)|$$

$$\geq 0$$

$$\Rightarrow \lambda = \operatorname{sign}(x \cdot \cos(\ln|x|+c))$$

B52

a) $$\int_2^3 \frac{dx}{4x^2-9} = \int_2^3 \frac{dx}{(2x-3)(2x+3)} = \int_2^3 \frac{A}{2x-3} + \frac{B}{2x+3}$$

Zuhaltemethode:

$2x-3 \to 0$

$$x \to 3/2 \quad A + B\cdot 0 = \frac{1}{2x+3}\Big|_{x=3/2} = 1/6$$

$$x \to 3/2 \quad A\cdot 0 + B = \frac{1}{2x-3}\Big|_{x=3/2} = -1/6$$

$$\int_2^3 \frac{1/6}{2x-3} - \frac{1/6}{2x+3} = \frac{1}{6}\cdot\frac{1}{2}\Big[\ln(2x-3) - \ln(2x+3)\Big]_2^3$$

$$= \frac{1}{12}\cdot\left(\ln(3) - \ln(1) - \ln(9) + \ln(7)\right) = \underline{\underline{\frac{1}{12}\left(\ln\left(\frac{7}{3}\right)\right)}}$$

b) $$\int_0^1 \frac{dx}{4x^2+8x+13} = \int_0^1 \frac{dx}{(2x+2)^2+9} \qquad v = \frac{1}{3}(2x+2)$$

$$= 3/2 \cdot 1/9 \int_{2/3}^{4/3} \frac{dx}{v^2+1} = 1/6\left(\arctan\frac{4}{3} - \arctan 2/3\right)$$

B53

a) $\int_3^4 \frac{dx}{\sqrt{x^2-9}} \underset{x=3\cosh x}{=} \int_0^{\text{arcosh } 4/3} \frac{3\sinh v}{3\sqrt{\cosh^2 v - 1}}\, dv$

$= \int_0^{\text{arcosh } 4/3} 1 \cdot dv = \text{arcosh } 4/3 = \ln(4+\sqrt{7}) - \ln 3$

$\frac{4}{3} = \cosh y = \frac{e^y + e^{-y}}{2}$

$\Rightarrow y = \log \frac{4 \pm \sqrt{7}}{3}$

b) $\int_2^3 \frac{3e^x + 4e^{-x} + 2}{1 - e^{2x}}\, dx \underset{y=e^x}{=} \int_{e^2}^{e^3} \frac{3y + \frac{4}{y} + 2}{(1-y^2)y}\, dy$

$= \int_{e^2}^{e^3} \frac{3y^2 + 2y + 4}{y^2(y+1)(y-1)}\, dy = \dots = \int_{e^2}^{e^3} \left(\frac{-9/2}{y-1} + \frac{5/2}{y+1} + \frac{4}{y^2} + \frac{2}{y} \right) dy$

$= -\frac{9}{2}\ln(e^3-1) + \frac{5}{2}\ln(e^3+1) - 4e^{-3} + 6 + \frac{9}{2}\ln(e^2-1) - \frac{5}{2}\ln(e^2+1)$

$+ 4e^{-2} \cancel{-4} + 2$

$$\int_{-2}^{2} \sqrt{|1-x^2|}\,dx = 2\cdot\int_{0}^{2}\sqrt{|1-x^2|}\,dx$$

$$= 2\left(\int_0^1 \sqrt{1-x^2}\,dx + \int_1^2 \sqrt{x^2-1}\,dx\right)$$

$\hookrightarrow u = \sin x$ $\quad\quad$ $\hookrightarrow w = \cosh x$

$$= \frac{\pi}{2} + 2\sinh(\cancel{\text{arcco}}\ \operatorname{arcosh} 2) - \operatorname{arcosh} 2$$

$$= \frac{\pi}{2} + e^{2+\sqrt{3}} - e^{-(2+\sqrt{3})} - (2+\sqrt{3})$$

c) $$\int_0^{1/\sqrt{2}} \frac{x^4}{\sqrt{1-x^2}}\,dx \underset{x=\sin y}{=} -\int_0^{\pi/4} \frac{\sin^4 y}{\cancel{|\cos y|}}\,\cancel{\cos y}\,dy$$

$$= -\int_0^{\pi/4} \sin^4 y\,dy$$

$$\int_0^{\pi/4} \sin^4 y\,dy = \int_0^{\pi/4} \sin^2 y\,(1-\cos^2 y)\,dy = \int_0^{\pi/4}\sin^2 y\,dy - \int_0^{\pi/4}\sin^2 y\cos y\cdot\cos y\,dy$$

$$= \int_0^{\pi/4}\sin^2 y\,dy - \left[\frac{1}{3}\sin^3 y\cdot\cos y\right]_0^{\pi/4} - \frac{1}{3}\sin^4 y\,dy$$

$$\Leftrightarrow \frac{4}{3}\int_0^{\pi/4}\sin^4 y\,dy = -\frac{1}{3}\left(\frac{1}{2}\sqrt{2}\right)^4 + \int_0^{\pi/4}\sin^2 y\,dy$$

$$\Rightarrow \int_0^{1/\sqrt{2}} \frac{x^4}{\sqrt{1-x^2}}\,dx = \frac{1}{4} - \frac{3\pi}{32}$$