Aufgabe A30

a) $$\int_0^{\pi/4} x\cdot\tan^2(x)\,dx = \Big[x\cdot\big(\tan(x)-x\big)\Big]_0^{\pi/4} - \int_0^{\pi/4} 1\cdot\big(\tan(x)-x\big)\,dx$$

denn $\int \tan^2(x)\,dx = (\tan(x)-x)+C$ (siehe S. 31 im Skript.)

$$= \frac{\pi}{4}\cdot\left(1-\frac{\pi}{4}\right) - 0 - \int_0^{\pi/4}\frac{\sin(x)}{\cos(x)}\,dx + \int_0^{\pi/4} x\,dx$$

$$= \frac{\pi}{4}\left(1-\frac{\pi}{4}\right) + \Big[\log(\cos(x))\Big]_0^{\pi/4} + \Big[\tfrac{1}{2}x^2\Big]_0^{\pi/4}$$

$\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + C$

$$= \underline{\underline{\frac{\pi}{4}\left(1-\frac{\pi}{4}\right) + \log\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{\pi}{4}\right)^2}}$$

b) $$\int_0^{2\pi}\sqrt{1+\cos(x)}\,dx =$$

$\left[t=\cos(x),\ x=\arccos(t),\ \frac{dx}{dt}=\frac{-1}{\sqrt{1-t^2}}\right.$

$$\int_0^{\pi}\sqrt{1+\cos(x)}\,dx + \int_{\pi}^{2\pi}\sqrt{1+\cos(x)}\,dx$$

Problem: $\cos(x)$ <u>nur</u> auf $[0,\pi)$ invertierbar

Lösung: Integral zerlegen!

$$= \int_0^{\pi}\sqrt{1+\cos(x)}\,dx + \int_{\pi}^{2\pi}\sqrt{1-\cos(x+\pi)}\,dx \qquad [y=x+\pi]\quad \frac{dy}{dx}=1$$

$$= \int_0^{\pi}\sqrt{1+\cos(x)}\,dx + \int_{2\pi}^{3\pi}\sqrt{1-\cos(y)}\,dy$$

$$= \int_0^{\pi}\sqrt{1+\cos(x)}\,dx + \int_0^{\pi}\sqrt{1-\cos(x)}\,dx$$

$$= \int_1^{-1}\sqrt{1+t}\cdot\frac{-1}{\sqrt{1-t^2}} + \sqrt{1-t}\cdot\frac{-1}{\sqrt{1-t^2}}\,dt$$

$$= \int_{-1}^{1}\frac{1}{\sqrt{1-t}} + \frac{1}{\sqrt{1+t}}\,dt = \Big[-2\sqrt{1-t} + 2\sqrt{1+t}\Big]_{-1}^{1} = \underline{\underline{2\sqrt{2}+2\sqrt{2}}}$$

Aufgabe A31

$$f(x) = \frac{x^8 - 2x^4 - x^2 + 5x - 1}{(x^2-1)^2} =: \frac{P_8(x)}{Q_4(x)}$$

1\. Schritt (Division mit Rest)

- $$\frac{P_8(x)}{Q_4(x)} = \ldots = S_4(x) + \frac{T_2(x)}{Q_4(x)}$$ mit $S_4(x) = x^4 + 2x^2 + 1$
  $T_2(x) = -x^2 + 5x - 2$

2\. Schritt (Nullstellen von Q_4 bestimmen und Q_4 zerlegen)

- $Q_4(x) = 0 \Leftrightarrow (x^2-1) = 0 \Leftrightarrow x = +1 \vee x = -1$
- $Q_4(x) = (x^2-1)^2 = (x-1)^2 (x+1)^2$

3\. Schritt (Partialbruchzerlegung)

3.1 (Überprüfen, ob T_2 und Q_4 gemeinsamen Nullstellen besitzen, und ggf. kürzen):

$T_2(1) \neq 0$ und $T_2(-1) \neq 0 \Rightarrow$ keine gemeinsamen Nullstelle

3.2 (Ansatz)

$$(*) \quad \frac{T_2(x)}{Q_4(x)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)} + \frac{D}{(x+1)^2}$$

3.3 (Koeffizienten bestimmen):

- $(x-1)^2 \frac{T_2}{Q_4} = A(x-1) + B + C \cdot \frac{(x-1)^2}{(x+1)} + D \cdot \frac{(x-1)^2}{(x+1)^2} \quad \forall x \in \mathbb{R} \setminus \{\pm 1\}$

$x \to 1 \Downarrow$

$$\frac{T_2(1)}{(1+1)^2} = 0 + B + 0 + 0 \Rightarrow \underline{\underline{B = \frac{1}{2}}}$$

- $(x+1)^2 \frac{T_2}{Q_4} = A \cdot \frac{(x+1)^2}{(x-1)} + B \frac{(x+1)^2}{(x-1)^2} + C \cdot (x+1) + D \quad \forall x \in \mathbb{R} \setminus \{\pm 1\}$

$x \to -1 \Downarrow$

$$\frac{T_2(-1)}{(-1-1)^2} = 0 + 0 + 0 + D = \underline{\underline{D = -2}}$$

- $x = 0$ in $(*)$ einsetzen $\Rightarrow \frac{T_2(0)}{Q_4(0)} = \frac{A}{-1} + \frac{B}{1} + \frac{C}{1} + \frac{D}{1}$

$$\Rightarrow \frac{-2}{1} = -A + \frac{1}{2} + C - 2$$

$$\Rightarrow A - C = \frac{1}{2} \quad (**)$$

- $x = 2$ in $(*)$ einsetzen $\Rightarrow \frac{T_2(2)}{Q_4(2)} = \frac{A}{1} + \frac{B}{1} + \frac{C}{3} + \frac{D}{9}$

$$\Rightarrow \frac{4}{9} = A + \frac{1}{2} + \frac{C}{3} - \frac{2}{9} \Rightarrow \frac{1}{6} = A + \frac{C}{3}$$

- $(**) \wedge (***) \Leftrightarrow A = \frac{1}{2} + C \wedge C = \frac{1}{2} - 3A \Leftrightarrow \underline{\underline{A = \frac{1}{4}}} \wedge \underline{\underline{C = -\frac{1}{4}}}$

Ergebnis: $f(x) = x^4 + 2x^2 + 1 + \frac{1/4}{(x-1)} + \frac{1/2}{(x-1)^2} + \frac{-1/4}{(x+1)} + \frac{-2}{(x+1)^2}$

b) $\int_0^{1/2} f(x)\,dx = \Big[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x + \frac{1}{4}\log|x-1| + \frac{1}{2}(-1)(x-1)^{-1} - \frac{1}{4}\log|x+1|$

$-2(-1)(x+1)^{-1}\Big]_0^{1/2}$

$$= \underline{\underline{\frac{1}{5}\frac{1}{2^5} + \frac{2}{3}\cdot\frac{1}{2^3} + \frac{1}{2} + \frac{1}{4}\log\frac{1}{2} + \frac{1}{2}\cdot 2 - \frac{1}{4}\log\left(\frac{3}{2}\right) + 2\cdot\frac{2}{3} - \frac{1}{2} - 2}}$$

Aufgabe A32 Sei $x \in (0,1)$.

a) $\displaystyle\int \frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}-1}\, dK \qquad \left[x = \sinh(t),\ \frac{dx}{dt} = \cosh(t) \quad 1+x^2 = \cosh^2(t) \right]$

$$= \int \frac{\cosh(t)+1}{\cosh(t)-1} \cosh(t)\, dt$$

$$= \int \frac{\frac{1}{2}(e^t+e^{-t})+1}{\frac{1}{2}(e^t+e^{-t})-1} \cdot \frac{1}{2}(e^t+e^{-t})\, dt$$

$$= \frac{1}{2} \int \frac{e^t+e^{-t}+2}{e^t+e^{-t}-2}\, e^t dt + \frac{1}{2} \int \frac{e^t+e^{-t}+2}{e^t+e^{-t}-2}\, e^{-t} dt$$

$$= \frac{1}{2} \int \frac{y+y^{-1}+2}{y+y^{-1}-2}\, dy - \frac{1}{2} \int \frac{z^{-1}+z+2}{z^{-1}+z-2}\, dz$$

$$\left[y = e^t,\ \frac{dy}{dt} = e^t \right] \qquad \left[z = e^{-t},\ \frac{dz}{dt} = -e^{-t} \right]$$

$$= \frac{1}{2} \int \frac{y^2+1+2y}{y^2+1-2y}\, dy - \frac{1}{2} \int \frac{1+z^2+2z}{1+z^2-2z}\, dz$$

$$= \frac{1}{2} \int 1 + \frac{4y}{y^2+1-2y}\, dy - \frac{1}{2} \int 1 + \frac{4z}{z^2+1-2z}\, dz$$

$$= \frac{1}{2} y - \frac{1}{2} z + \frac{1}{2} \int \frac{4(y-1)+4}{(y-1)^2}\, dy - \frac{1}{2} \int \frac{4(z-1)+4}{(z-1)^2}\, dz$$

$$= \frac{1}{2}(y-z) + \frac{1}{2}\int \frac{4}{(y-1)}\,dy + \frac{1}{2}\int \frac{4}{(y-1)^2}\,dy - \frac{1}{2}\int \frac{4}{(z-1)}\,dz - \frac{1}{2}\int \frac{4}{(z-1)^2}\,dz$$

$$= \frac{1}{2}(y-z) + 2\log|y-1| + (-2)(y-1)^{-1} - 2\log|z-1| - (-2)(z-1)^{-1} + C$$

$$= \frac{1}{2}(e^{t}-e^{-t}) + 2\log|e^{t}-1| - 2\cdot\frac{1}{(e^{t}-1)} - 2\log|e^{-t}-1| + \frac{2}{(e^{-t}-1)} + C$$

$$= x + 2\log|e^{\operatorname{arsinh}(x)}-1| - \frac{2}{(e^{\operatorname{arsinh}(x)}-1)}$$

$$- 2\log|e^{-\operatorname{arsinh}(x)}-1| + \frac{2}{(e^{-\operatorname{arsinh}(x)}-1)} + C \,, \quad C\in\mathbb{R}$$

b) $\displaystyle\int \frac{1}{\sqrt{1-x^2}+1-x}\,dx \qquad \left[x=\sin(t),\ \underset{\substack{\uparrow \\ \text{ok für } x\in(0,1)}}{t=\arcsin(x)},\ \frac{dx}{dt}=\cos(t)\right.$

$$= \int \frac{1}{\underbrace{\sqrt{1-\sin^2(t)}}_{=|\cos t|}+1-\sin(t)}\cos(t)\,dt$$

$$= \int \frac{\cos(t)}{\cos(t)+1-\sin(t)}\,dt \qquad ;\ x\in(0,1) \Rightarrow t\in\left(-\tfrac{\pi}{2},\tfrac{\pi}{2}\right) \Rightarrow \cos(t)>0$$

$$= \int \frac{\left(\frac{1-y^2}{1+y^2}\right)}{\left(\frac{1-y^2}{1+y^2}\right)+1-\left(\frac{2y}{1+y^2}\right)}\left(\frac{2}{1+y^2}\right)dy$$

Halbwinkelmethode

$$\left[\underset{t=2\arctan(y)}{y=\tan\left(\tfrac{t}{2}\right)},\ \frac{dt}{dy}=\frac{2}{1+y^2},\ \cos(t)=\frac{1-y^2}{1+y^2},\ \sin(t)=\frac{2y}{1+y^2}\right]$$

$$= \int \frac{1-y^2}{1-y^2+1+y^2-2y}\cdot\left(\frac{2}{1+y^2}\right)dy$$

$$= \int \frac{1-y^2}{2(1-y)}\cdot\frac{2}{1+y^2}\,dy = \int \frac{1+y}{1+y^2}\,dy$$

$$= \int \frac{1}{1+y^2}\,dy + \frac{1}{2}\int \frac{2y}{1+y^2}\,dy$$

$$= \arctan(y) + \frac{1}{2}\log(1+y^2) + C, \quad C\in\mathbb{R}$$

$$= \frac{1}{2}\arcsin(x) + \frac{1}{2}\log\left(1+\tan^2\left(\tfrac{1}{2}\arcsin(x)\right)\right) + C, \quad C\in\mathbb{R}$$

Aufgabe A33 Es sei $x \in (-3,-1)$, $x \neq -2$.

$$\int \frac{\sqrt{-x^2-4x-3}}{(x+2)^2}\,dx = \int \frac{\sqrt{1-(x+2)^2}}{(x+2)^2}\,dx$$

$$\left[\; x+2 = \cos(t),\quad \frac{dx}{dt} = -\sin(t) \;\right]$$

OK für $x \in (-3,-1) \Leftrightarrow t \in (0,\pi)$

$$= -\int \frac{\overbrace{\sqrt{1-\cos^2(t)}}^{=|\sin(t)|}}{\cos^2(t)}\,\sin(t)\,dt$$

$\sin(t) > 0$ auf $(0,\pi)$

$$= -\int \frac{\sin^2(t)}{\cos^2(t)}\,dt$$

$$= -\int \tan^2(t)\,dt$$

$$= t - \tan(t) + C, \quad C \in \mathbb{R}$$

$$= \arccos(x+2) - \tan(\arccos(x+2)) + C, \quad C \in \mathbb{R}$$