Mathe 12B

http://themaex.de/files/HM2.rar

B54.

a) $$\int_2^\infty \frac{1}{x^2+4x-7}\,dx$$

$$\left|\frac{1}{x^2+4x-7}\right| = \frac{1}{x^2+4x-7} \le \frac{1}{x^2} \quad \text{für } x \ge \frac{7}{4}$$

$$\int_1^\infty x^\alpha\,dx \begin{cases} < \infty \\ = \infty \end{cases} \iff \alpha \begin{cases} < -1 \\ \ge -1 \end{cases}$$

$$\int_2^\infty \frac{1}{x^2}\,dx \quad \text{konvergiert}$$

$\Rightarrow$ Majorantenkriterium

$$\Rightarrow \int_2^\infty \frac{1}{x^2+4x-7}\,dx \quad \text{konvergiert}$$

b) $$\int_1^\infty \frac{x^3+3x\cos x-\sin x}{4x^4+2\ln x}\,dx$$

für $x \ge 3$

$$x^3+3x\cos x-\sin x \ge x^3-3x-1 \ge 0$$

dann: $$\frac{x^3+3x\cos x-\sin x}{4x^4+2\ln x} \ge \frac{x^3 \overset{-3x-1}{\cancel{+3x\cos x-\sin x}}}{4x^4+2x^4}$$

$\ln x \le x^2$

$$\geq \frac{\cancel{1}\frac{1}{2}x^3}{6x^4} = \frac{1}{12x}$$

$$\int_1^\infty \frac{1}{12} \cdot \frac{1}{x}$$ divergiert

$\Rightarrow$ Minorantenkriterium $\Rightarrow \int_1^\infty \dots dx$ divergiert auch

c) $$\int_1^\infty \frac{\cos(3x)}{\sqrt{x}}\, dx$$

$$= \frac{1}{3} \left. \frac{\sin 3x}{\sqrt{x}} \right|_1^d + \frac{1}{6} \int_1^d \frac{\sin 3x}{\sqrt{x}^3}\, dx$$

$$\frac{1}{6} \left| \frac{\sin 3x}{\sqrt{x}^3} \right| \leq \cancel{\frac{1}{6}} \frac{1}{\sqrt{x}^3} = x^{-3/2}$$

$\Rightarrow$ konvergiert

$\Rightarrow$ Majorante

$\Rightarrow$ ..

$$\lim_{d\to\infty} \left(\left[\frac{\sin 3x}{\sqrt{x}} \right]_1^d \right) = \underbrace{\lim_{d\to\infty} \left(\frac{\sin 3d}{\sqrt{d}} \right)}_{\to 0} - \frac{\sin 3}{1}$$

$\Rightarrow \int_1^\infty \frac{\cos 3x}{\sqrt{x}}\, dx$ konvergiert

Alternativ Lemma 10.7. ③

$f: [a, \infty) \to \mathbb{R} \quad a>0$

$F(x) = \int_a^x f(t)\, dt$ sei beschränkt $\forall x \in [a, \infty)$

$\Rightarrow \int_a^\infty f(t)\, t^{-\alpha}\, dt$ ~~beschränkt~~ konvergiert für $\alpha > 0$

d) $\int_0^\pi \frac{1}{\sqrt{\sin x}}\, dx$

$\int_0^{\pi/2} \frac{1}{\sqrt{\sin x}}\, dx$

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

Kleinwinkelnäherung

$\Rightarrow$ für $x << 1$

$\frac{1}{2}x \leq \sin x < 2x$

$\int_0^1 x^\alpha\, dx \begin{cases} < \infty \\ = \infty \end{cases} \Leftrightarrow \alpha \begin{cases} > -1 \\ \leq -1 \end{cases}$

$\Rightarrow \frac{1}{\sqrt{\sin x}} \leq \sqrt{\frac{2}{x}}$

$\int_0^1 \frac{1}{x}\, dx$ konvergiert, $\underbrace{\int_0^{\pi/2} \frac{1}{\sqrt{x}}\, dx}_{\text{Majorante}}$, $\int_0^{\pi/2} \frac{1}{\sqrt{\sin x}}\, dx$ ebenso

$\frac{1}{\sqrt{\sin x}}$ symmetrisch um $\pi/2$

$$\Rightarrow \infty > 2 \cdot \int_0^{\pi/2} \frac{1}{\sqrt{\sin x}}\,dx = \int_0^{\pi} \frac{1}{\sqrt{\sin x}}\,dx \rightarrow \text{konvergiert}$$

e) $\int_0^\infty \frac{\arctan e^x}{x^2+1}\,dx$

$\arctan x < \frac{\pi}{2} \quad \forall x \in \mathbb{R}$

$$\left| \frac{\arctan e^x}{x^2+1} \right| = \frac{\arctan e^x}{x^2+1} \leq \frac{\pi/2}{x^2+1}$$

$$\Rightarrow \int_0^d \frac{\pi}{2} \frac{1}{x^2+1}\,dx = \frac{\pi}{2} \cdot \left[\arctan x\right]_0^d$$

$$\xrightarrow{d\to\infty} \frac{\pi}{2} \cdot \frac{\pi}{2} < \infty$$

$$\Rightarrow \int_0^\infty \frac{\arctan e^x}{x^2+1}\,dx \text{ konvergiert}$$

f) $\int_1^\infty \sin^2\left(\frac{1}{x}\right) dx$

$$\lim_{x\to\infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = 1$$

$\frac{1}{2x} \leq \sin\left(\frac{1}{x}\right) \leq \frac{2}{x}$ für x groß

$\rightarrow \sin^2 \frac{1}{x} \leq \left(\frac{2}{x}\right)^2 = \frac{4}{x^2}$

$\int_1^\infty \frac{1}{x^2}\, dx$ konvergiert

Majorante $\rightarrow$ $\int_1^\infty \sin^2\left(\frac{1}{x}\right) dx$ konvergiert

g) $\int_0^\infty \frac{e^x}{x^2+e^x}\, dx$

$\geq \int_0^\infty \frac{e^x}{e^x+e^x}\, dx = \int_0^\infty \frac{1}{2}\, dx$ divergiert

$\rightarrow$ Minorante $\rightarrow$ $\int_0^\infty \frac{e^x}{x^2+e^x}$ divergiert

B56

$\int_0^\infty \frac{x^\alpha}{1+x^\beta}\, dx$

$\beta \geq 0 \qquad x^\beta \geq 1$

$\int_1^\infty \frac{x^\alpha}{1+x^\beta} \geq \int_1^\infty \frac{x^\alpha}{\frac{1}{2}x^\beta}$

$\frac{1}{2} x^{\alpha-\beta} \leq \frac{x^\alpha}{1+x^\beta} \leq x^{\alpha-\beta}$

Majorante $x^{\alpha-\beta}$ für $\alpha < \beta-1$

$\alpha \geq \beta - 1$

Minorante

$\beta < 0$ $\qquad$ $(-\beta) > 0$

$$\frac{x^\alpha}{1+x^\beta} = \frac{x^{-\beta}}{x^{-\beta}} \cdot \frac{x^\alpha}{1+x^\beta} = \frac{x^{\alpha-\beta}}{1+x^{-\beta}}$$

$$\Rightarrow \frac{1}{2}x^\alpha \leq \frac{x^{\alpha-\beta}}{1-x^{-\beta}} \leq x^\alpha$$

$$\int_0^\infty \frac{x^\alpha}{1+x^\beta}\,dx = \begin{cases} < \infty & \text{falls } \alpha < -1, \beta < 0 \ \vee\ \alpha < \beta - 1, \beta > 0 \\ = \infty & \text{falls } \alpha \geq -1, \beta < 0 \ \vee\ \alpha \geq \beta - 1, \beta > 0 \end{cases}$$

b) $\int_0^\infty e^{-x^\alpha}\,dx$

$\alpha > 0$ $:= \frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)$ $\qquad$ 154 → konvergiert

$$\frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right) = \frac{1}{\alpha}\int_0^\infty e^{-t}\, t^{1/\alpha - 1}\,dt$$

$$= \int_0^\infty e^{-t}\left(t^{1/\alpha}\right)'\,dt \qquad t = x^\alpha$$

$$= \int_0^\infty e^{-x^\alpha}\,dx$$

$\alpha < 0$: $\int_0^\infty e^{-x^\alpha}\,dx = \int_0^\infty e^{-\frac{1}{x^{|\alpha|}}}\,dx$

$\Rightarrow$ für $x \geq 1$:

$\Rightarrow \frac{1}{x^{|\alpha|}} \leq 1 \quad \Rightarrow \frac{-1}{x^{|\alpha|}} \geq -1 \quad \Rightarrow e^{-\frac{1}{x^{|\alpha|}}} \geq \frac{1}{e}$

$\Rightarrow \int_0^\infty \frac{1}{e} \, dx$ divergiert

$\Rightarrow \int_0^\infty e^{-x^\alpha} dx$ divergiert

c) $\int_1^\infty \frac{(\ln x)^\alpha}{x^\beta} dx$

$\gamma > 0$

$\lim_{x \to \infty} \frac{\ln x}{x^\gamma}$

$\lim_{x \to \infty} \frac{1}{x} \cdot \frac{1}{\gamma x^{\gamma - 1}} = \lim_{x \to \infty} \frac{1}{\gamma} \frac{1}{x^\gamma} = 0$

$\Rightarrow$ für x groß: $\ln x \leq x^\gamma \quad \forall \gamma > 0$

Beh: konvergent für $\beta > 1 \quad \alpha \in \mathbb{R}$

Bew: sei $\alpha > 0 \quad \beta > 1$ Wähle $0 < \varepsilon < \frac{\beta - 1}{\alpha}$

$\ln x < x^\varepsilon$ für $x > k$

$\Rightarrow \frac{(\ln x)^\alpha}{x^\beta} \leq \frac{x^{\varepsilon\alpha}}{x^\beta} = x^{\varepsilon\alpha - \beta}$

$\varepsilon\alpha - \beta < \frac{\beta - 1}{\alpha} \alpha - \beta = -1$

$\Rightarrow$ konvergiert

$\alpha < 0$:

$(\ln x)^\alpha \leq 1$ für $x \geq e$

$\Rightarrow x^{-\beta}$

$\rightarrow$ konvergiert

<u>Beh:</u> Integral divergiert für $\beta \leq 1$

$\ln(x+1) \geq \frac{x}{x+1}$ für $x \geq 0$

$$\Rightarrow \frac{(\ln x)^\alpha}{x^\beta} = \frac{\ln((x-1)+1)^\alpha}{x^\beta} \geq \left(\frac{x-1}{x}\right)^\alpha \cdot \frac{1}{x^\beta}$$

$$= \frac{(x-1)^\alpha}{x^{\alpha+\beta}} \geq \frac{\left(\frac{x}{2}\right)^\alpha}{x^{\alpha+\beta}} \quad \text{für } x \geq 2$$

$$= \frac{1}{2^\alpha} \cdot \frac{1}{x^\beta}$$

$\int_1^\infty \frac{1}{x^\beta}\,dx$ divergiert

$\Rightarrow$ <u>Beh</u>